Proceedings of Basic and Applied Sciences
ISSN 1857-8179 (paper). ISSN 1857-8187 (online). http://www.anglisticum.mk
Proceedings of the 1st International Conference on New Horizons in Basic and Applied Science, Hurghada – Egypt, Vol 1(1), 2013.

Estimation of Stress-Strength Parameter for Burr Type XII Distribution Based on Progressive Type-II Censoring

A. M.Abd-Elfattah[1], M. H. Abu-Moussa[2*]
[1]Department of Mathematical Statistics, Institute of Statistical Studies-Research Cairo University, Egypt.
[2]Department of Mathematics, Faculty of Science, Cairo University, Giza-Egypt.
*E-mail of the Corresponding author: mhmoussa@sci.cu.edu.eg

Received November 13, 2013; Revised November 14, 2013; Accepted November 15, 2013

Abstract: *In this paper,the estimation of stress-strength parameter $R = P(Y < X)$ is considered When X, Y the strength and stress respectively are two independent random variables of Burr Type XII distribution. The samples taken for X and Y are progressively censoring of type II. The maximum likelihood estimator (MLE) of R is obtained when the common parameter is unknown. But when the common parameter is known the MLE, uniformly minimum variance unbiased estimator (UMVUE) and the Bayes estimator of $R = P(Y < X)$ are obtained. The exact confidence interval of R based on MLE is obtained. Also the performance of the proposed estimators is compared using the computer simulation.*

Keywords: *Burr Type XII distribution; progressive type-II censoring; stress-strength model; unbiased estimator; maximum-likelihood estimator; uniformly minimum variance unbiased estimator; confidence intervals; Bayes estimator*

Introduction

In 1942 I.W. Burr[1] published a system of cumulative distribution functions (cdfs) that might be useful "for purposes of graduation", he has suggested twelve types. Special attention has been devoted to the type XII and type X in modeling lifetime data or survival data. The Burr Type XII has the following distribution function for $X > 0$:

$$F(x; p, b) = 1 - (1 + x^b)^{-p}; \qquad p > 0, b > 0 \tag{1.1}$$

And the density function of Burr Type XII for $X > 0$ denoted by BurrXII(p, b) is

$$f(x; p, b) = pbx^{b-1}(1 + x^b)^{-(p+1)}; \qquad p > 0, b > 0 \tag{1.2}$$

Burr Type XII distribution has different special cases of life time distributions, one of them is the Weibull distribution when $p = \infty$. In life-testing experiments, one often encounters situations where it takes a substantial amount of time to obtain a reasonable number of failures necessary to carry out reliable inference, so censored samples are used for analyzing lifetime data. Among various censoring schemes, the Type II progressive censoring scheme has become very popular one in the last decade. It can be described as follows: let n units be placed on test at time zero with m failures to be observed. At the first failure a number r_1 of the surviving units $(n-1)$ are randomly selected and removed from the experiment. At the second observed failure, r_2 of the surviving units $(n-r_1-2)$ are randomly selected and removed from the experiment, and so on until the m-th failure is observed. The all remaining surviving units $r_m = n-m-r_1-r_2-...-r_{m-1}$ are removed.We denote to progressively Type II censoring with scheme $(n, m, r_1, r_2, ..., r_m)$. Traditional Type II censoring scheme is included when $(r_1 = r_2 = ... = r_{m-1} = 0)$ and $(r_m = n-m)$ and complete sampling scheme when $(n = m)$ and $(r_1 = ... = r_{m-1} = r_m = 0)$. Balakrishnan and Aggarwala[2] and Balakrishnan[3] present a study on different features of progressive censoring schemes.

In stress-strength model, the stress (Y) and the strength (X) are treated as random variables and the reliability of a component during a given period is taken to be the probability that its strength exceeds the stress during the entire interval, i.e. the reliability R of a component is $R = P(Y < X)$. For a particular situation, if we consider Y as the pressure of a chamber generated by ignition of a solid propellant and X as the strength of the chamber. Then R represents the probability of successful firing of the engine. Stress-strength model can be used as a general measure of the difference between two populations and has applications in many area. For example comparing two treatments X and Y, then $R = P(Y < X)$ is the measure of the response of treatment X. For other applications see Kotz et al.[4]. Many authors considered the problem of estimating the stress-strength parameter based on complete samples, it first considered by Birnbaum[5]. Johnson[6] present a good review on stress-strength model in reliability. Awad and Charraf [7] studied the case when X and Y are independent Burr random variables of type XII, they obtained maximum likelihood, uniformly minimum unbiased (MVUE) and Bayesian estimates of R. Ahmed et al. [8] consider this problem when X and Y are two independent random variables have Burr Type X distribution. Based on censored samples Saraçoğlu et al.[9] obtained the estimation for R based on exponential distribution with type II progressive censoring. Abd-Elfattah et al.[10] get the estimation of R based on Weibull distribution with type II progressive censoring, they discussed two cases the first when X and Y have common shape parameter and different scale parameters while the second case when X and Y have common scale parameter and different shape parameters. For some of the recent references, the readers may refer to [11-13].

In the present paper, the study the estimation of $R = P(Y < X)$ when X and Y are two independent but not identically random variables belonging to burr type XII distribution with two parameters. In Section (2), maximum likelihood estimator of reliability R is obtained in two subsections first when the common parameter b is unknown while the second when b is known. UMVUE of R and Bayes estimator when b is known are obtained in sections (3) and (4) respectively. Numerical results using simulations are presented in Sections (5).Some concluding remarks given in section (6).

ISSN 1857-8179 (paper). ISSN 1857-8187 (online). http://www.anglisticum.mk
Proceedings of the 1st International Conference on New Horizons in Basic and Applied Science, Hurghada – Egypt, Vol 1(1), 2013.

MLE of R

In this section the MLE of R is obtained. Let X and Y are two independent Burr Type XII random variables with parameters (p,b) and (q,b) then R is:

$$R = P(Y < X) = \int_0^\infty \int_0^x f(x) f(y) dy dx$$

$$= \int_0^\infty \int_0^x pbx^{b-1}(1+x^b)^{-(p+1)}.qby^{b-1}(1+y^b)^{-(q+1)} dydx \quad = \frac{q}{p+q} \tag{2.1}$$

So we deal with two cases when the common parameter b is unknown and known which are mentioned in the following subsections.

If common parameter b is unknown

Let $X_{1:m_1:n_1},....,X_{m_1:m_1:n_1}$ be a progressive censored sample from BurrXII(p,b) with progressive censoring scheme $(n_1, m_1, r_1,..., r_{m_1})$, and let $Y_{1:m_2:n_2},....,Y_{m_2:m_2:n_2}$ be a progressive censored sample from BurrXII(q,b) with progressive censoring scheme $(n_2, m_2, s_1,..., s_{m_2})$, then the jointly likelihood function L(p,q,b) is

$$L(p,q,b) = [k_1 \prod_{i=1}^{m_1} f(x_i)[1-F(x_i)]^{r_i}].[k_2 \prod_{j=1}^{m_2} f(y_j)[1-F(y_j)]^{s_j}] = k_1 k_2 p^{m_1} q^{m_2} b^{m_1+m_2} \prod_{i=1}^{m_1} x_i^{b-1} \prod_{j=1}^{m_2} y_j^{b-1} \prod_{i=1}^{m_1} (1+x_i^b)^{-p-1-pr_i} \prod_{j=1}^{m_j} (1+y_j^b)^{-q-1-qs_j} \tag{2.2}$$

Where k_1 and k_2 are:

$k_1 = n_1(n_1 - 1 - r_1)(n_1 - 2 - r_1 - r_2)...(n_1 - m_1 + 1 - r_1 - ...r_{m_1-1})$ $k_2 = n_2(n_2 - 1 - s_1)(n_2 - 2 - s_1 - s_2)...(n_2 - m_2 + 1 - s_1 - ...s_{m_2-1})$ (2.3) Now the log-likelihood function ℓ is:

$$\ell = lnk_1k_2 + m_1 lnp + m_2 lnq + (m_1 + m_2) lnb + (b-1)\sum_{i=1}^{m_1} lnx_i$$

$$+ (b-1)\sum_{j=1}^{m_2} lny_j - \sum_{i=1}^{m_1} (1 + p(1+r_i)) ln(1+x_i^b) \tag{2.4}$$

$$- \sum_{j=1}^{m_2} (1 + q(1+s_j)) ln(1+y_j^b)$$

By differentiation on equation (2.4) with respect to p, q and b, and setting the results equal to zero. Then we get:

$$\frac{\partial \ell}{\partial p} = \frac{m_1}{p} - \sum_{i=1}^{m_1} (1+r_i) ln(1+x_i^b) = 0 \tag{2.5}$$

$$\frac{\partial \ell}{\partial q} = \frac{m_2}{q} - \sum_{j=1}^{m_2} (1+s_j) ln(1+y_j^b) = 0 \tag{2.6}$$

$$\frac{\partial \ell}{\partial b} = \frac{m_1 + m_2}{b} + \sum_{i=1}^{m_1} lnx_i + \sum_{j=1}^{m_2} lny_j - \sum_{i=1}^{m_1}(1 + p(1+r_i))\frac{x_i^b lnx_i}{(1+x_i^b)} \quad (2.7)$$
$$-\sum_{j=1}^{m_2}(1+q(1+s_j))\frac{y_j^b lny_j}{(1+y_j^b)}$$

From equations (2.5),(2.6) and (2.7), we get

$$\hat{p} = m_1\left[\sum_{i=1}^{m_1}(1+r_i)ln(1+x_i^{\hat{b}})\right]^{-1} \quad (2.8)$$

$$\hat{q} = m_2\left[\sum_{j=1}^{m_2}(1+s_j)ln(1+y_j^{\hat{b}})\right]^{-1} \quad (2.9)$$

We can obtain $\hat{b}$ by solving the following non-linear equation:

$$\hat{b} = (m_1+m_2)\left[-\sum_{i=1}^{m_1} lnx_i - \sum_{j=1}^{m_2} lny_j + \left[\frac{m_1}{\sum_{i=1}^{m_1} ln(1+x_i^{\hat{b}})} + 1\right]\sum_{i=1}^{m_1}\frac{x_i^{\hat{b}} lnx_i}{(1+x_i^{\hat{b}})} + \left[\frac{m_2}{\sum_{j=1}^{m_2} ln(1+y_j^{\hat{b}})} + 1\right]\sum_{j=1}^{m_2}\frac{y_j^{\hat{b}} lny_j}{(1+y_j^{\hat{b}})}\right]^{-1} \quad (2.10)$$

This equation can be solved numerically using Newton Rhapson Method with initial values closed to real values of parameters. Then MLE of R is

$$\hat{R} = \frac{\hat{q}}{\hat{p}+\hat{q}} \quad (2.11)$$

If common parameter b is known

Assume b is known, then without loss of generality we can assume that $b = 1$. Then let $X_{1:m_1:n_1},....,X_{m_1:m_1:n_1}$ be a progressive censored sample from BurrXII(p,1) with progressive censoring scheme $(n_1, m_1, r_1,..., r_{m_1})$, and let $Y_{1:m_2:n_2},....,Y_{m_2:m_2:n_2}$ be a progressive censored sample from BurrXII(q,1) with progressive censoring scheme $(n_2, m_2, s_1,..., s_{m_2})$. Then from equations (2.8), (2.9)

$$\hat{p} = m_1\left[\sum_{i=1}^{m_1}(1+r_i)ln(1+x_i)\right]^{-1} \quad (2.12) \qquad \hat{q} = m_2\left[\sum_{j=1}^{m_2}(1+s_j)ln(1+y_j)\right]^{-1} \quad (2.13)$$

Therefore $$\hat{R} = \frac{m_2\sum_{i=1}^{m_1}(1+r_i)ln(1+x_i)}{m_1\sum_{j=1}^{m_2}(1+s_j)ln(1+y_j) + m_2\sum_{i=1}^{m_1}(1+r_i)ln(1+x_i)} \quad (2.14)$$

Now consider $U = 2p\sum_{i=1}^{m_1}(1+r_i)ln(1+x_i) : \chi^2_{2m_1}$ and $V = 2q\sum_{j=1}^{m_2}(1+s_j)ln(1+y_j) : \chi^2_{2m_2}$

Then

$$\hat{R} = \frac{U}{U + \frac{2m_1 p}{2m_2 q} V} = \frac{1}{1 + \frac{p}{q} F} \qquad (2.15)$$

Where $F = \frac{V/2m_2}{U/2m_1} : F(2m_2, 2m_1)$. Hence

$$\frac{R}{1-R} \times \frac{1-\hat{R}}{\hat{R}} = F : F(2m_2, 2m_1) \; (2.16)$$

Then the $100(1-\alpha)\%$ exact confidence interval of R is:

$$P\left(\frac{1}{1 + F_{2m_1,2m_2,\alpha/2}(\frac{1}{\hat{R}} - 1)} < R < \frac{1}{1 + F_{2m_1,2m_2,1-\alpha/2}(\frac{1}{\hat{R}} - 1)} \right) = 1 - \alpha \; (2.17)$$

Where α is the level of significance and $2m_1, 2m_2$ are the degree of freedom of F.

UMVUE of R

In this section the uniformly minimum variance unbiased estimator (UMVUE) is obtained for stress-strength parameter R. Let $X_{1:m_1:n_1}, \ldots, X_{m_1:m_1:n_1}$ be a progressive censored sample from BurrXII(p,b) with progressive censoring scheme $(n_1, m_1, r_1, \ldots, r_{m_1})$, assuming the common parameter b is known. The log-likelihood function of X is:

$$ln\, L = lnk_1 + m_1 lnb + m_1 lnp + (b-1)\sum_{i=1}^{m_1} lnx_i - \sum_{i=1}^{m_1} ln(1 + x_i^b) - p\sum_{i=1}^{m_1}(1 + r_i) ln(1 + x_i^b) \qquad (3.1)$$

Where k_1 mentioned in equation (2.3). Then from equation (3.1) we obtained that $\sum_{i=1}^{m_1}(1+r_i) ln(1+x_i^b)$ is a sufficient statistics for p. Similarly for the progressive censored sample $Y_{1:m_2:n_2}, \ldots, Y_{m_2:m_2:n_2}$ from BurrXII(q,b) with progressive censoring scheme $(n_2, m_2, s_1, \ldots, s_{m_2})$, we obtained that $\sum_{j=1}^{m_2}(1+s_j) ln(1+y_j^b)$ is a sufficient statistics for q. Let $T_i = ln(1 + X_i^b)$, consider the following transformations:

$$Z_1 = n_1 T_1$$
$$Z_2 = (n_1 - r_1 - 1)[T_2 - T_1]$$
$$Z_{m_1} = (n_1 - r_1 - r_2 - \ldots - r_{m_1-1} - m_1 + 1)[T_{m_1} - T_{m_1-1}] \; (3.2)$$

Balakrishnan & Aggarwala [2] show that $Z_i^{;s}$ are independent & identically distributed exponential random variables with mean p,

moreover $T = \sum_{i=1}^{m_1} Z_i = \sum_{i=1}^{m_1}(1+r_i)T_i = \sum_{i=1}^{m_1}(1+r_i)ln(1+X_i^b)$ (3.3)

Then T has a gamma distribution with shape parametereter m_1 and scale parameter p with probability density function:

$$f_T(t) = \frac{1}{p^{m_1}\Gamma(m_1)} t^{m_1-1} exp(\frac{-t}{p}), \qquad 0 < t < \infty \quad (3.4)$$

Lemma 3.1 *The conditional p.d.f. of* $T_1 = ln(1+X_1^b)$ *given* T *is:*

$$f_{T_1|T}(x) = \frac{f_{T_1,T}(x)}{f_T(t)} = n_1(m_1-1)\frac{(T-n_1T_1)^{m_1-2}}{T^{m_1-1}}, \qquad 0 < T_1 < T/n_1 \quad (3.5)$$

Proof. Let $W = \sum_{i=2}^{m_1} Z_i$ then clearly W & Z_1 are independent. Then the joint p.d.f. of T_1 & T, $f_{T_1,T}(x)$ can be easily obtained from the jointly distribution of W & Z_1 using the transformations $Z_1 = n_1T_1$ & $W = T - Z_1$ then

$$f_{W,Z_1} = f_W \cdot f_{Z_1} = \frac{1}{p^{m_1}\Gamma(m_1-1)} W^{m_1-2} exp(-\frac{W+Z_1}{p}) \quad (3.6)$$

And

$$f_{T_1,T} = \frac{n_1}{p^{m_1}\Gamma(m_1-1)}(T-n_1T_1)^{m_1-2} exp(-\frac{T}{p}) \quad (3.7)$$

From equations (3.7),(3.4), we get the result.

Similarly if $E = \sum_{j=1}^{m_2}(1+s_j)E_j$ where $E_j = ln(1+Y_j^b)$ Then

$$f_{E_1|E}(y) = n_2(m_2-1)\frac{(E-n_2E_1)^{m_2-2}}{E^{m_2-1}}, \qquad 0 < E_1 < E/n_2 \quad (3.8)$$

Lemma 3.2 *The unbiased estimator of R is:*

$$\phi(T_1, E_1) = \begin{cases} 1 & if \quad n_2E_1 < n_1T_1 \\ 0 & if \quad n_2E_1 \geq n_1T_1 \end{cases} \quad (3.9)$$

Where $E_1 = ln(1+Y_1^b)$ and $T_1 = ln(1+X_1^b)$.

Proof.

$$E(\phi) = 1.P(n_2E_1 < n_1T_1) = P(Y_1 < [(1+X_1^b)^{n_1/n_2} - 1]^{1/b})$$
$$= P(Y_1 < a) = \int_0^\infty\int_0^a f_{X_1}(x) f_{Y_1}(y)dydx \qquad (3.10)$$

Where the distributions of order statistics X_1 and Y_1 are

$$f_{X_1}(x) = n_1 pbx^{b-1}(1+x^b)^{-pn_1-1}$$
$$f_{Y_1}(y) = n_2 qby^{b-1}(1+y^b)^{-qn_2-1} \qquad (3.11)$$

Then by using equations(3.11) we get

$$E(\phi) = \frac{q}{p+q} = R \qquad (3.12)$$

Theorem 3.3 *Based on the sufficient statistics T and E, as defined before for p and q respectively and the unbiased statistics* ϕ*, the UMVUE of* R*, say* $\tilde{R}$*, for* $m_1 \geq 2$ *and*

Selection and peer-review under responsibility of ICNHBAS, 2013 http://www.nhbas2013.com

$m_2 \geq 2$ *can be expressed as follows:*

$$\tilde{R} = \begin{cases} 1 - \sum_{k=0}^{m_2-1} (-1)^k \left(\frac{T}{E}\right)^k \frac{\binom{m_2-1}{k}}{\binom{m_1+k-1}{k}} & if \quad T < E \\ 1 - \sum_{k=0}^{m_1-1} (-1)^k \left(\frac{E}{T}\right)^k \frac{\binom{m_1-1}{k}}{\binom{m_2+k-1}{k}} & if \quad T \geq E \quad (3.13) \end{cases}$$

Proof.

For $T < E$ using the Rao-Blackwell theorem

$\tilde{R} = E(\phi(T_1, E_1) \mid T, E) = \iint_A f_{(T_1|T)} f_{(E_1|E)} dE_1 dT_1,$ (3.14)

Where $A = \{(E_1, T_1): 0 < T_1 < \frac{T}{n_1}, 0 < E_1 < \frac{E}{n_2}\}$ and $n_2 E_1 < n_1 T_1\}$ and $f_{(T_1|T)}$ & $f_{(E_1|E)}$ are defined in equations(3.5),(3.8)respectively. Then $\tilde{R}$ becomes:

$$\tilde{R} = \int_0^{T/n_1} \int_0^{n_1 T_1/n_2} n_1(m_1-1) \frac{(T-n_1T_1)^{m_1-2}}{T^{m_1-1}} . n_2(m_2-1) \frac{(E-n_2E_1)^{m_2-2}}{E^{m_2-1}} dE_1 dT_1$$

let $c = \frac{n_1 T_1}{T}$, then $\tilde{R}$ becomes:

$$= 1 - \int_0^{T/n_1} n_1(m_1-1) \frac{(T-n_1T_1)^{m_1-2}}{T^{m_1-1}} \frac{(E-n_1T_1)^{m_2-1}}{E^{m_2-1}} dT_1 \qquad (3.15)$$

$$\tilde{R} = 1 - \int_0^1 (m_1-1)(1-c)^{m_1-2} \left(1 - c\frac{T}{E}\right)^{m_2-1} dc \qquad (3.16)$$

Since the binomial expansion of $(1 - c\frac{T}{E})^{m_2-1} = \sum_{k=0}^{m_2-1} (-1)^k \binom{m_2-1}{k} (\frac{cT}{E})^k$. Then $\tilde{R}$ is obtained as following:

$$\tilde{R} = 1 - \sum_{k=0}^{m_2-1} (-1)^k \binom{m_2-1}{k} \left(\frac{T}{E}\right)^k \int_0^1 c^k (1-c)^{m_1-2} dc \quad = 1 - \sum_{k=0}^{m_2-1} (-1)^k \left(\frac{T}{E}\right)^k \frac{\binom{m_2-1}{k}}{\binom{m_1+k-1}{k}} \qquad (3.17)$$

If $T \geq E$ then $\tilde{R}$ becomes:

$$\tilde{R} = 1 - \sum_{k=0}^{m_1-1} (-1)^k \left(\frac{E}{T}\right)^k \frac{\binom{m_1-1}{k}}{\binom{m_2+k-1}{k}} \qquad (3.18)$$

Bayes Estimator of R

In this section the Bayes estimator of R is obtained when the parameters p and q are random variables. For both populations of X and Y we assume that the common parameter b is known. Now assume we have the Gamma priors for p and q with the following probability density functions

$$\pi(p) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} p^{\alpha_1-1} e^{-\beta_1 p}, \qquad p > 0 \quad (4.1)$$

And $\pi(q)=\dfrac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}q^{\alpha_2-1}e^{-\beta_2 q}, \qquad q>0$ (4.2) Here $\alpha_1,\beta_1,\alpha_2$ and $\beta_2>0$. Let $X_1,\dots,X_{m_1}$ be a progressive censoring sample of X, the Likelihood function of X is:

$$L_1(p)=f(x_1,\dots,x_{m_1}\mid p)=k_1 p^{m_1}b^{m_1}\prod_{i=1}^{m_1}x_i^{b-1}\times\prod_{i=1}^{m_1}(1+x_i^b)^{-p-1-pr_i} \qquad (4.3)$$

Where k_1 is defined in equation(2.3). Now to find the posterior distribution we should find the marginal distribution of X,

$$f(x_1,\dots,x_{m_1})=\int_0^\infty f(x_1,\dots,x_{m_1}\mid p)\pi(p)dp$$

$$=\frac{k_1 b^{m_1}\beta_1^{\alpha_1}\prod_{i=1}^{m_1}x_i^{b-1}\prod_{i=1}^{m_1}(1+x_i^b)^{-1}}{[\beta_1+\sum_{i=1}^{m_1}(1+r_i)ln(1+x_i^b)]^{m_1+\alpha_1}}*\frac{\Gamma(m_1+\alpha_1)}{\Gamma(\alpha_1)} \qquad (4.4)$$

Then the posterior distribution

$$\pi_1(p\mid x_1,\dots,x_{m_1})=\frac{f(x_1,\dots,x_{m_1}\mid p)\pi(p)}{f(x_1,\dots,x_{m_1})}$$

$$\pi_1(p\mid x_1,\dots,x_{m_1})=\frac{\lambda_1^{m_1+\alpha_1}}{\Gamma(m_1+\alpha_1)}p^{m_1+\alpha_1-1}e^{-p\lambda_1} \qquad (4.5)$$

Which mean that $\pi_1(p\mid x_1,\dots,x_{m_1})\sim Gamma(m_1+\alpha_1,\lambda_1)$, where $\lambda_1=\beta_1+\sum_{i=1}^{m_1}(1+r_i)ln(1+x_i^b)$

Similarly for censored sample $y_1,\dots,y_{m_2}$ the posterior function of q is:

$$\pi_2(q\mid y_1,\dots,y_{m_2})=\frac{\lambda_2^{m_2+\alpha_2}}{\Gamma(m_2+\alpha_2)}q^{m_2+\alpha_2-1}e^{-q\lambda_2} \qquad (4.6)$$

$\pi_2(q\mid y_1,\dots,y_{m_2})\sim Gamma(m_2+\alpha_2,\lambda_2)$,

and $\lambda_2=\beta_2+\sum_{j=1}^{m_2}(1+s_j)ln(1+y_j^b)$

Both p and q are independent then we can find the joint posterior function of p and q:

$$\pi(p,q\mid x,y)=H.p^{m_1+\alpha_1-1}q^{m_2+\alpha_2-1}e^{-p\lambda_1-q\lambda_2} \qquad (4.7)$$

Where $H=\dfrac{\lambda_1^{m_1+\alpha_1}\lambda_2^{m_2+\alpha_2}}{\Gamma(m_1+\alpha_1)\Gamma(m_2+\alpha_2)}$. Let $r=\dfrac{q}{p+q}$ and $\xi=p+q$

where $0<r<1,\xi>0$,then

$$\pi(r,\xi\mid x,y)=H.\xi^{m_1+m_2+\alpha_1+\alpha_2-1}r^{m_2+\alpha_2-1}(1-r)^{m_1+\alpha_1-1}\times exp(-\xi[r\lambda_2+(1-r)\lambda_1]) \qquad (4.8)$$

Integrate out ξ $\pi(r\mid x,y)=H.r^{m_2+\alpha_2-1}(1-r)^{m_1+\alpha_1-1}\dfrac{\Gamma(m_1+m_2+\alpha_1+\alpha_2)}{[r\lambda_2+(1-r)\lambda_1]^{m_1+m_2+\alpha_1+\alpha_2}}, \quad 0<r<1$ (4.9)

Using equation(4.9), Bayes estimator of R, say $\hat{R}_{BS}$, under squared error loss function is

$$\hat{R}_{BS} = E(R \mid x, y) = \int_0^1 r\pi(r \mid x, y)dr \quad \hat{R}_{BS} = H.\Gamma(m_1+m_2+\alpha_1+\alpha_2)\int_0^1 \frac{r^{m_2+\alpha_2}(1-r)^{m_1+\alpha_1-1}}{[r\lambda_2+(1-r)\lambda_1]^{m_1+m_2+\alpha_1+\alpha_2}}dr$$

$$= (m_2+\alpha_2)\Gamma(m_1+m_2+\alpha_1+\alpha_2)\lambda_1^{-m_2-\alpha_2}\lambda_2^{m_2+\alpha_2}$$

$$\times F_{2,1}[1+m_2+\alpha_2, m_1+m_2+\alpha_1+\alpha_2, 1+m_1+m_2+\alpha_1+\alpha_2, 1-\frac{\lambda_2}{\lambda_1}] \tag{4.10}$$

The final form of $\hat{R}_{BS}$ in equation(4.10)is calculated using Mathematica program.

Simulation Study

Within this section, the Monte Carlo simulation is performed to check the performance of the different estimators of R under several types of progressive censoring schemes. Samples are generated under progressive type-II censoring with many different schemes for the (n-m) removed items. This schemes are described as follows:

Scheme I: complete sample (n=m)i.e there is no removed items.

Scheme II: $(r_1 = 0,..., r_{m-1} = 0, r_m = n-m)$.

Scheme III: $(r_1 = n-m,..., r_{m-1} = 0, r_m = 0)$.

Scheme IV: The remaining items (n-m)are removed equally at each failure time. For example if n=10 and m=5 then scheme IV become $(r_1 = 1, r_2 = 1,..., r_5 = 1)$.

Different values of parameters $(b, p, q) = (1,10,5),(1,10,8)$ are used. Simulation is performed 1000 times with different sample sizes $n_1, n_2 = 10,20,30$ and the number of failures $m_1, m_2 = 5,10,15,20,30$ for X and Y. The average estimates of MLE for R in case of b is unknown and average MSE's are reported is Table 1. Also the MLE, UMVUE and 95% exact confidence interval of R when b is known are obtained and the average estimates and average MSE's are reported is Table 2, 3. Also simulation is constructed 1000 times for Baysian estimator of R suggested in Section (4), and the averages of estimates and MSE's are reported in Table 4 with the following configurations for the parameters of priors of p,q : $\alpha_1 = 0,1,2,20$, $\alpha_1 = 0,1,2,20$ and $\beta_1 = 0,1.$, $\beta_2 = 0,1$. We note that in such cases as the effective sample size increases the estimates of R become better. When $n = m$ i.e in case of complete samples the biased is decreased. Also when $(n_1, m_1) = (n_2, n_2)$ the estimates are good. We note that MLE of R give results better than the UMVUE of R and Bayes estimator. Bayes estimator depend on the prior parameters of p,q. We note that the results become better when the values of $\alpha_1, \alpha_2, \beta_1$ and β_2 tends to zero, and when α_1, α_2 greater than β_1, β_2 as in case of $(\alpha_1, \beta_1) = (20,0)$ and $(\alpha_2, \beta_2) = (20,0)$.

Conclusion

We have presented some efficient estimators of the stress-strength parameter R using MLE, UMVUE and Bayes estimator methods. The methods are very efficient. We have found that, our estimates of R using progressive censoring schemes are very close to estimates in case of complete samples so this estimates are better to accelerate the life testing. This work gives a general estimates since the case when sample sizes equal the number of failures is a special case. The exact confidence intervals of R based on MLE when parameter b is known are obtained. Choice of sample sizes and number of failures are affect on the estimates. Also choosing the hyper parameter values of priors distributions of p and q affect on the Bayes estimates. We note that MLE is more effective than the other methods. Numerical results are presented which exhibit the performance of the proposed methods.

References

1. Burr, I.W., “Cumulative frequency distribution,” Annals of Mathematical Statistics vol. 13, 215-232, 1942.
2. Balakrishnan, N. and Aggarwala, R., Progressive Censoring: Theory, Methods, and Applications. Birkhäuser, Boston, 2000.
3. Balakrishnan, N., "Progressive censoring methodology: an appraisal", (with discussions)", TEST, vol.16, no. 2, 211 - 296, 2007.
4. Kotz, S., Lumelskii, Pensky, M., The stress-strength model and its generalization: theory and applications, World Scientific, Singapore, 2003.
5. Birnbaum, Z.W., On a use of the Mann-Whitney statistic. Proceedings of the Third Berkeley Symposium Math. Statistic Probability I, 13-17, 1956.
6. Johnson, R. A., Stress strength models for reliability. Handbook of statistics, P. R. Krishaiah and C. R. Rao, Amsterdam, North-Holland, 1988.
7. Awad, A.M. and Charraf, M.K., Estimation of $P(Y < X)$ in the Burr case; A comparative study. Commun. Statist. Simul.,15(2), 389-403, 1986.
8. Ahmed, K. E., Fakhry, M. E. and Jaheen, Z. F., Empirical Bayes estimation of P(Y<X) and characterizations of Burr-type X model. Journal of Statistical Planning and Inference, 64(2), 297-308, 1997.
9. Saraço $\breve{g}$ lu, B, Kinaci, I and Kundu, D, on estimation of $R = P(Y < X)$ for exponential distribution under progressive Type –II censoring, Journal of Statistics Computation and Simulation, 729-744, vol. 82, 5, 2012.
10. Abd-Elfattah, A.M., El-Fahham and M. M., Abu-Moussa, M. H., Estimation of Stress-Strength Parameter for Weibull Distribution Based on Progressive Type-II Censoring, Journal of Advanced Research, 2013. (submitted)
11. Abd-Elfattah, A.M., and R.M. Mandouh, ‘Estimation of PrY < X in Lomax case,” The 39th Annual Conference on Statistics, Computer Science and Operation Research, ISSR, Cairo University, Egypt, part 1, 156-166, 2004.

12. Panahi, H., and S. Asadi. "Estimation of $R = P[Y < X]$ for two-parameter Burr Type XII Distribution." World Academy of Science, Engineering and Technology 72, 465-470, 2010.
13. Asgharzadeh, A., Valiollahi, R., & Raqab, M. Z. Stress-strength reliability of Weibull distribution based on progressively censored samples. Sort: Statistics and Operations Research Transactions, 35(2), 103-124, 2011.
14. Zimmer, W.J., Keats, J.B., Wang, F.K.(1998). The Burr XII distribution in reliability analysis, Journal of Quality Technology, 30, 386-394.
15. Lie, Y.L., Tsai, T.R., Wu, S.J. (2010). Acceptance sampling plans from truncated life tests based on the Burr type XII percentiles, Journal of the Chinese Institute of Industrial Engineers, 27:4, 270-280.
16. Rao, G. Srinivasa, Muhammad Aslam, and Debasis Kundu. "Burr-XII Distribution Parametric Estimation and Estimation of Reliability of Multicomponent Stress-Strength."

ISSN 1857-8179 (paper). ISSN 1857-8187 (online). http://www.anglisticum.mk
Proceedings of the 1st International Conference on New Horizons in Basic and Applied Science, Hurghada – Egypt, Vol 1(1), 2013.

Appendix

Table 1: MLE and MSE for R when $b=1$ is unknown and $p=10$ and $q=5,8$

(n_1,m_1)	(n_2,m_2)	r	s	$q=5$ $\hat{R}_1$ MSE	$q=8$ $\hat{R}_1$ MSE
(10,10)	(10,10)	Comp	Comp	0.332087 $1.27\,.10^{-2}$	0.438199 $1.53\,.10^{-2}$
(10,5)	(10,5)	II	II	0.300286 $3.95\,.10^{-2}$	0.420466 $4.69\,.10^{-2}$
		III	III	0.301881 $1.65\,.10^{-2}$	0.438423 $1.95\,.10^{-2}$
		IV	IV	0.333221 $1.39\,.10^{-2}$	0.450485 $1.61\,.10^{-2}$
(20,20)	(10,10)	Comp	Comp	0.33834 $9.10\,.10^{-3}$	0.445884 $1.01\,.10^{-2}$
(20,10)	(10,5)	II	II	0.290334 $3.05\,.10^{-2}$	0.441256 $3.56\,.10^{-2}$
		III	III	0.308726 $1.45\,.10^{-2}$	0.448033 $1.85\,.10^{-2}$
		IV	IV	0.356571 $1.14\,.10^{-2}$	0.469956 $1.24\,.10^{-2}$
(10,5)	(20,10)	II	II	0.271972 $3.04\,.10^{-2}$	0.40147 $3.69\,.10^{-2}$
		III	III	0.268575 $1.66\,.10^{-2}$	0.410227 $1.93\,.10^{-2}$
		IV	IV	0.306936 $9.94\,.10^{-3}$	0.412853 $1.32\,.10^{-2}$
(20,10)	(20,10)	II	II	0.27807 $1.98\,.10^{-2}$	0.424797 $2.75\,.10^{-2}$
		III	III	0.274073 $1.29\,.10^{-2}$	0.417762 $1.38\,.10^{-2}$
		IV	IV	0.33009 $5.74\,.10^{-3}$	0.443401 $6.18\,.10^{-3}$
(20,20)	(20,20)	Comp	Comp	0.335086 $5.83\,.10^{-3}$	0.442428 $6.61\,.10^{-3}$
(30,15)	(30,15)	II	II	0.273742 $1.57\,.10^{-2}$	0.416884 $1.79\,.10^{-2}$
		III	III	0.258116 $1.27\,.10^{-2}$	0.415254 $1.14\,.10^{-2}$
		IV	IV	0.330451 $3.92\,.10^{-3}$	0.445149 $4.63\,.10^{-3}$
(30,30)	(30,30)	Comp	Comp	0.328542 $3.81\,.10^{-3}$	0.444942 $4.16\,.10^{-3}$

Table 2: MLE, UMVUE and MSE for R and Exact 95% C.I when $b=1$ is known, $p=10$ and $q=5$

(n_1,m_1)	(n_2,m_2)	r	s	MLE $\hat{R}_1$ MSE	UMVUE $\hat{R}_2$ MSE	Confidence Interval Lower Upper
(10,10)	(10,10)	Comp	Comp	0.341639 $9.77\,.10^{-3}$	0.326304 $7.45\,.10^{-3}$	0.173936 0.561188
(10,5)	(10,5)	II	II	0.348302 $2.01\,.10^{-2}$	0.302093 $1.35\,.10^{-2}$	0.125717 0.665154
		III	III	0.343207 $1.06\,.10^{-2}$	0.31473 $9.41.10^{-3}$	0.125062 0.663822
		IV	IV	0.342593 $1.10\,.10^{-2}$	0.313735 $9.86\,.10^{-3}$	0.122968 0.659507
(20,20)	(10,10)	Comp	Comp	0.345988 $8.10\,.10^{-3}$	0.433341 $4.31\,.10^{-2}$	0.155492 0.522415

(20,10)	(10,5)	II	II	0.356239 $1.65.10^{-2}$	0.396423 $4.28.10^{-2}$	0.139321 0.605503
		III	III	0.353362 $8.36.10^{-3}$	0.410174 $4.43.10^{-2}$	0.137821 0.602496
		IV	IV	0.365609 $9.91.10^{-3}$	0.386814 $4.15.10^{-2}$	0.144264 0.615164
(10,5)	(20,10)	II	II	0.328087 $1.32.10^{-2}$	0.101488 $5.72.10^{-2}$	0.149691 0.62536
		III	III	0.329393 $7.57.10^{-3}$	0.094183 $5.84.10^{-2}$	0.150446 0.626746
		IV	IV	0.315545 $7.58.10^{-3}$	0.0904713 $6.02.10^{-2}$	0.142522 0.611801
(20,10)	(20,10)	II	II	0.34131 $9.96.10^{-3}$	0.325438 $8.32.10^{-3}$	0.173726 0.560827
		III	III	0.334279 $5.04.10^{-3}$	0.328497 $5.27.10^{-3}$	0.16926 0.553072
		IV	IV	0.339186 $5.21.10^{-3}$	0.327921 $4.99.10^{-3}$	0.172372 0.558496
(20,20)	(20,20)	Comp	Comp	0.337798 $4.79.10^{-3}$	0.333638 $4.68.10^{-3}$	0.213856 0.4889
(30,15)	(30,15)	II	II	0.333471 $6.68.10^{-3}$	0.329519 $5.81.10^{-3}$	0.194351 0.50923
		III	III	0.336635 $3.40.10^{-3}$	0.329509 $3.41.10^{-3}$	0.196584 0.512779
		IV	IV	0.334756 $3.30.10^{-3}$	0.331923 $3.41.10^{-3}$	0.195257 0.510674
(30,30)	(30,30)	Comp	Comp	0.335202 $3.29.10^{-3}$	0.332761 $3.32.10^{-3}$	0.23225 0.456646

Table 3: MLE, UMVUE and MSE for R and Exact 95% C.I when $b=1$ is known, $p=10$ and $q=8$

(n_1,m_1)	(n_2,m_2)	r	s	MLE	UMVUE	Confidence Interval
				$\hat{R}_1$ MSE	$\hat{R}_2$ MSE	Lower Upper
(10,10)	(10,10)	Comp	Comp	0.44677 $1.21.10^{-2}$	0.393109 $7.99.10^{-3}$	0.246807 0.665578
(10,5)	(10,5)	II	II	0.451453 $2.22.10^{-2}$	0.356652 $1.68.10^{-2}$	0.177303 0.748568
		III	III	0.445173 $1.21.10^{-2}$	0.389865 $9.24.10^{-3}$	0.177548 0.748884
		IV	IV	0.445062 $1.29.10^{-2}$	0.385025 $9.91.10^{-3}$	0.177482 0.748799
(20,20)	(10,10)	Comp	Comp	0.448648 $9.21.10^{-3}$	0.255676 $5.44.10^{-2}$	0.220704 0.62722
(20,10)	(10,5)	II	II	0.459537 $1.72.10^{-2}$	0.264047 $6.42.10^{-2}$	0.199181 0.702235
		III	III	0.461852 $1.10.10^{-2}$	0.23626 $6.29.10^{-2}$	0.200672 0.70418
		IV	IV	0.478139 $1.11.10^{-2}$	0.226394 $6.60.10^{-2}$	0.211366 0.717617
(10,5)	(20,10)	II	II	0.435119 $1.70.10^{-2}$	0.161434 $9.28.10^{-2}$	0.217352 0.724764
		III	III	0.430027 $9.31.10^{-3}$	0.146309 $9.31.10^{-2}$	0.213843 0.720606
		IV	IV	0.42254 $1.02.10^{-2}$	0.138839 $9.68.10^{-2}$	0.208741 0.714401
(20,10)	(20,10)	II	II	0.435119 $1.70.10^{-2}$	0.394872 $8.21.10^{-3}$	0.217352 0.724764
		III	III	0.44507 $6.48.10^{-3}$	0.419475 $4.06.10^{-3}$	0.24553 0.664045
		IV	IV	0.447023 $6.47.10^{-3}$	0.417857 $4.08.10^{-3}$	0.246997 0.665806

(20,20)	(20,20)	Comp	Comp	0.445494 6.04 $.10^{-3}$	0.422511 3.56.10^{-3}	0.299935 0.601045
(30,15)	(30,15)	II	II	0.446913 7.65 $.10^{-3}$	0.411772 4.99.10^{-3}	0.280374 0.626282
		III	III	0.448107 4.07 $.10^{-3}$	0.427901 2.77.10^{-3}	0.28135 0.627412
		IV	IV	0.445901 4.08 $.10^{-3}$	0.428054 2.75.10^{-3}	0.279549 0.625323
(30,30)	(30,30)	Comp	Comp	0.446244 4.06 $.10^{-3}$	0.431388 2.46.10^{-3}	0.325907 0.57323

Table 4: Bayes estimator of R, $\hat{R}_3$ and MSE when $b=1$ is known, $p=10$ and $q=5$

(α_1,β_1)	(α_2,β_2)	r	s	(n_1,n_2,m_1,m_2)=(20,20,10,10)	(n_1,n_2,m_1,m_2)=(30,30,15,15)
				$\hat{R}_3$ MSE	$\hat{R}_3$ MSE
(0,0)	(0,0)	II	II	0.345916 9.68 $.10^{-3}$	0.344215 6.18 $.10^{-3}$
		III	III	0.343206 4.88 $.10^{-3}$	0.340206 3.19 $.10^{-3}$
		IV	IV	0.34443 5.02 $.10^{-3}$	0.341563 3.14 $.10^{-3}$
(0,1)	(0,1)	II	II	0.405455 8.53 $.10^{-3}$	0.392077 6.66 $.10^{-3}$
		III	III	0.389718 5.79 $.10^{-3}$	0.373264 3.87 $.10^{-3}$
		IV	IV	0.386531 5.56 $.10^{-3}$	0.370808 3.65 $.10^{-3}$
(1,1)	(1,1)	II	II	0.40355 8.54 $.10^{-3}$	0.391856 6.38 $.10^{-3}$
		III	III	0.392241 6.16 $.10^{-3}$	0.373716 3.83 $.10^{-3}$
		IV	IV	0.386204 5.49 $.10^{-3}$	0.371241 3.78 $.10^{-3}$
(2,1)	(2,1)	II	II	0.409699 9.57 $.10^{-3}$	0.394493 6.86 $.10^{-3}$
		III	III	0.389637 5.86 $.10^{-3}$	0.377316 4.13 $.10^{-3}$
		IV	IV	0.388655 5.76 $.10^{-3}$	0.370916 3.76 $.10^{-3}$
(20,0)	(20,0)	II	II	0.341576 9.82 $.10^{-3}$	0.341557 6.84 $.10^{-3}$
		III	III	0.336704 4.99 $.10^{-3}$	0.338465 3.60 $.10^{-3}$
		IV	IV	0.335965 5.34 $.10^{-3}$	0.338803 3.15 $.10^{-3}$
(α_1,β_1)	(α_2,β_2)	r	s	(n_1,n_2,m_1,m_2)=(20,20,20,20)	(n_1,n_2,m_1,m_2)=(30,30,30,30)
				$\hat{R}_3$ MSE	$\hat{R}_3$ MSE
(0,0)	(0,0)	Comp	Comp	0.340161 4.50 $.10^{-3}$	0.338312 3.16 $.10^{-3}$
(0,1)	(0,1)	Comp	Comp	0.38044 4.92 $.10^{-3}$	0.36557 3.28 $.10^{-3}$
(1,1)	(1,1)	Comp	Comp	0.380773 4.97 $.10^{-3}$	0.367009 3.37 $.10^{-3}$
(2,1)	(2,1)	Comp	Comp	0.378671 4.92 $.10^{-3}$	0.365881 3.20 $.10^{-3}$
(20,0)	(20,0)	Comp	Comp	0.337958 4.83 $.10^{-3}$	0.336623 3.40 $.10^{-3}$

Selection and peer-review under responsibility of ICNHBAS, 2013 http://www.nhbas2013.com