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The Evolution of the MacWilliams Extension Theorem to Codes over Finite Modules

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Abstract: *In the 1960's Florence MacWilliams proved two important results in coding theory. The first result is that all linear codes over finite fields satisfy the MacWilliams identities. The second result is the MacWilliams Extension Theorem. This theorem proved the equivalence of two notions of code equivalence over finite fields with respect to Hamming weight.*
This paper studies the evolution of the extension theorem from the classical case of linear codes defined over finite fields, to the case of linear codes defined over finite rings and finally to linear codes defined over finite modules.

Keywords: *Codes over modules, MacWilliams, extension theorem, Frobenius.*

Introduction

Coding theory lies in the intersection of three disciplines, mathematics, computer science and engineering. Research in this area is both application-driven, as in the case of computer science and engineering, or is theoretical, as in the case of mathematics. Of the points that have grabbed the attention of all three disciplines is the work of Florence MacWilliams. In the 1960's MacWilliams proved two fundamental results in coding theory, the MacWilliams identities and the MacWilliams Extension Theorem. The MacWilliams identities are a tool for studying the weight distributions of codes and have many applications in coding theory. The MacWilliams Extension Theorem gains its importance from the fact that it provides a method of identifying when two codes are equivalent. MacWilliams proved her results for codes defined over finite fields. This paper discusses the evolution of the MacWilliams Extension Theorem as the definition of a code was generalized from being over a finite field, to being over a finite ring, and eventually to being over a finite module.

Codes over Finite Fields

Codes were originally defined over finite fields, i.e. the code words used finite fields as their alphabet. In the following, F_q denotes the finite field containing q elements, where q is necessarily a prime power. We start by giving a formal definition of a linear code over a finite field and defining a weight function on these codes. In computer science, the most widely used field is F_2 which produces what is known as *binary* codes.

Definition 2.1 **(Linear Code over F_q)**

A linear code C of length n over the finite field F_q is a subspace of the vector space F_q^n. An element $c=c_0\, c_1 \ldots c_{n-1}$ in C is called a codeword.

Definition 2.2 **(Weight)**

A weight defined on the finite field F_q is a function w: $F_q \rightarrow \mathbb{Q}$ with w(0)=0. This function is then extended naturally to a weight on F_q^n by $w(x_1 \ldots x_n) = \sum_{i=1}^{n} w(x_i)$.

The most commonly used weight is the Hamming weight defined as follows,

Definition 2.3 **(Hamming Weight)**

Let C be a linear code of length n over F_q, then the Hamming weight of a codeword $c=c_0\, c_1 \ldots c_{n-1}$ in C is denoted by wt(c) and it is the number of nonzero entries in the vector c, in other words, $wt(c)=|\{i: c_i \neq 0, 0 \leq i \leq n-1\}|$.

Definition 2.4 **(Preserve Weight)**
A function f: $F_q^n \rightarrow F_q^n$ is said to preserve the weight w if w(x) = w(f(x)) for every $x \in F_q^n$.

It is important to know when two codes are considered to be *essentially* the same. The natural definition of the equivalence of two codes is the existence of a linear isomorphism between them that preserves Hamming weight. This is known as an isometry.

Definition 2.5 **(Isometry)**

Let C_1 and C_2 be two codes of length n over F_q^n of the same dimension. We say that $\varphi: C_1 \rightarrow C_2$ is an isometry if φ is a one-to-one linear transformation that preserves Hamming weight. If there is an isometry from C_1 to C_2, we say that the two codes are isometric.

Another notion of equivalence also existed, defining two codes to be *monomially equivalent* if there is a monomial transformation taking one code to the other; a monomial transformation basically permutes and re-scales the coordinates of each code word. In fact the restriction of such a transformation to the code gives a linear, Hamming weight preserving, isomorphism, so that any two codes that are monomially equivalent must also be isometric.

Definition 2.6 **(Monomial Equivalence)**
Let C_1 and C_2 be two codes of length n over F_q^n of the same dimension. Then C_1 and C_2 are monomially equivalent if there exists a monomial n x n matrix A (a matrix with only one nonzero entry in each row and column) such that $C_2 = \{xA: x \in C_1\}$.

Note that in the field F_2, monomial equivalence and permutation equivalence have the same meaning. In other words two codes of length n over F_2 are monomially equivalent if and only if we can obtain one code from the other via a permutation matrix, which permutes the coordinates of each code word.

Given a linear, Hamming weight preserving isomorphism between two codes, is it possible to extend this isomorphism to a monomial transformation of all of F_q^n? In the early 1960's Florence MacWilliams proved that it is always possible to find this extension and therefore the two notions of equivalence are in fact the same. This result is known as the "The MacWilliams Extension Theorem" or "The MacWilliams Equivalence Theorem".

Theorem 2.1 **(MacWilliams Extension Theorem)**

Two linear codes of length n are isometric if and only if they are monomially equivalent.

MacWilliams proved this theorem in her doctoral dissertation [11]. In 1978 another proof was presented by Bogart, Goldberg and Gordon in [1]; their proof utilized the vector space structure of a finite field and its relation to matrices. A different technique of proof was also provided by Ward and Wood in [13] using the linear independence of characters. Wood later generalizes this character theoretic proof to linear codes defined over finite modules.

Since a code over a finite field is a linear space, it is natural to speak of its dual.

Definition 2.7 **(Dual Code)**

Given a linear code C of length n over the field F_q, its dual code is defined as
$C^{\perp} := \{x \in C: < x, c > = 0, \text{for every } c \in C\}$.

An important aspect of a code is the Hamming weight of its code words and the number of code words having a certain Hamming weight. These elements come into play in the definition of the Hamming Weight Enumerator.

Definition 2.8 (Hamming Weight Enumerator)
The Hamming weight enumerator of a linear code C of length n is given by,
$W_C(X,Y) := \sum_{x \in C} X^{n-wt(x)} Y^{wt(x)} = \sum_{j=0}^{n} A_j X^{n-j} Y^j$,

where A_j denotes the number of code words in C of Hamming weight j.

Credit is owed Florence MacWilliams for another vital result in coding theory. In the 1960's MacWilliams proved that codes over finite fields satisfy the MacWilliams identities. These identities are equations that relate the Hamming weight enumerator of a code to that of its dual.

***Theorem 2.2* (MacWilliams Identities)**

Suppose C is a linear code of length n over a finite field F_q. The Hamming weight enumerators of C and its dual $C^\perp$ satisfy the following, $W_{C^\perp}(X,Y) = \frac{1}{|C|} W_C(X + (q-1)Y, X - Y)$.

The following theorem, relates linear codes over finite fields to their respective duals. The next theorem together with the MacWilliams identities were to become the model theorem for future generations of researchers who worked with, the more general, codes over rings and codes over modules.

Theorem 2.3

Suppose C is a linear code of length n over a finite field F_q. The dual code $C^\perp$ satisfies:

1. $C^\perp$ is a subset of F_q^n,
2. $C^\perp$ is a linear code of length n,
3. $(C^\perp)^\perp = C$,
4. $\dim(C^\perp) = n - \dim(C)$.

Codes over Finite Rings

Although known beforehand, interest in codes over finite rings began in the early 1990's with the work of Hammons, Calderbank, Sloane, Kumar and Solé. They published their findings in [8]; the authors discovered that certain well known non-linear binary codes can be constructed as the image of linear codes over the ring $\mathbb{Z}_4$. As interest in this area of research was renewed, mathematicians started to ask what types of codes over finite rings satisfy the crucial results of coding theory, such as the MacWilliams Extension Theorem and the MacWilliams identities.

***Definition 3.1* (Linear Code over R)**

Let R be a finite ring. A right (resp. left) linear code C is a right (resp. left) submodule C of R^n. Note that unlike the case of finite fields, R^n is not necessarily a vector space.

Two types of rings prove to be quite interesting in relation to coding theory. These are quasi-Frobenius and Frobenius rings. Recall that Artinian rings are rings that satisfy the descending chain condition on ideals.

Definition 3.2 **(Quasi-Frobenius Ring)**

A ring R is quasi-Frobenius if R is Artinian and self-injective, i.e. injective as a module over itself.

Recall the definition of the Jacobson radical of a ring, rad(R), which is the intersection of all maximal right(or left) ideals of R. Also recall that the socle of R, soc(R), is defined to be the sum of all the minimal right submodules of R. These two concepts are needed to define Frobenius rings.

Definition 3.3 **(Frobenius Ring)**

A ring R is Frobenius if $soc({}_R R) \cong {}_R(R/\,rad(R))$ and $soc(R_R) \cong (R/\,rad\,(R))_R$.

In the search for answers concerning the types of rings satisfying the MacWilliams Extension Theorem and MacWilliams identities, Wood singled out finite Frobenius rings in [14]. He generalized the character theoretic proof of the extension theorem over finite fields in [13] to prove the theorem for Frobenius rings. He followed Gleason's proof of the MacWilliams identities, using the Fourier Transform and Poisson summation formula to relate the weight enumerators of a code and a notion of its dual, namely the character theoretic annihilator. So given a linear code C over the ring R, the annihilator $(\hat{R}^n : C) := \{\varpi \in \hat{R} : \varpi(c) = 0, \text{ for every } c \in C\}$, where $\hat{R}$ is the ring of characters of R (all homomorphisms from R to $\mathbb{Q} / \mathbb{Z}$), acts as the dual of C in that it satisfies the MacWilliams identities and MacWilliams' model theorem.

Theorem 3.1 **(Extension Theorem)**

Let R be a finite Frobenius ring. Suppose C is a right linear code of length n over R, and suppose $f: C \to R^n$ is a right linear homomorphism which preserves Hamming weight. Then f extends to a right monomial transformation of R^n.

Greferath and Schmidt [7] gave another proof of this result using combinatorial results. Wood was also able to prove a partial converse in [14], in the case of finite commutative rings, where the definitions of quasi-Frobenius rings and Frobenius rings coincide.

Theorem 3.2

Suppose R is a finite commutative ring, and suppose that the extension theorem holds over R with respect to Hamming weight. Then R is Frobenius.

The question remained if the full converse of the extension theorem is true; i.e. any finite ring for which the extension theorem holds with respect to Hamming weight must necessarily be Frobenius. In 1997, Wood gave an example of a non-quasi-Frobenius ring that does not satisfy the theorem. Researchers attempted to prove that any finite ring for which the extension theorem holds must be quasi-Frobenius. But another result was found in 2000 in [7], where Greferath and Schmidt gave a counterexample of a quasi-Frobenius ring that did not satisfy the theorem. These results implied that the full converse of Wood's extension theorem may in fact be true.

As the search continued to try and prove the full converse of Wood's Extension Theorem, mathematicians were also considering general weight functions opposed to only the Hamming weight. Wood was able to prove that if R is a Frobenius ring, then the extension theorem holds with respect to symmetrized weight compositions [15]. The converse is still not known.

***Definition 3.4* (Symmetrized Weight Composition)**

Let G be a subgroup of the automorphism group of a finite ring R. Define $\sim$ on R by a $\sim$ b if and only if $a=\tau(b)$ for some $\tau \in G$. Let R/G denote the equivalence classes of this relation. The symmetrized weight composition is a function swc: $R^n \times R/G \rightarrow \mathbb{Q}$ defined by, $swc_r(x) = |\{i : x_i \sim r\}|$, where $x=(x_0,\ldots, x_{n-1}) \in R^n$ and $r \in R$.

All extension properties proven for the Hamming weight apply to homogeneous weights and vice versa as can be found in the works of Greferath, Schmidt, Nechaev and Wisbauer in [4], [6] and [7]. The definition of a homogeneous weight follows.

***Definition 3.5* (Homogeneous Weight)**

Let R be a finite ring and w be a weight defined on R. We say that w is homogeneous if:

1. all units u of R satisfy the following, w(ur)=w(r) for every r in R,
2. there exists a rational number γ such that $\sum_{s \in S} w(s) = \gamma|S|$, for every nonzero subring S of R.

A broader question presented itself, if R is a finite ring and w a weight on R, are there conditions on w to guarantee that the extension theorem will hold for codes over R with respect to the weight w? In [15], the author found a condition on weights defined on a special case of rings, namely, finite chain rings. A chain ring is a ring whose ideals form a chain. Note that finite chain rings are Frobenius. A weight w on a ring R is said to be maximally symmetric if it has the property that all automorphisms of R preserve w and that all units u of R satisfy w(ur)=w(r) for every r in R.

Theorem 3.3

Let R be a finite chain ring and w a maximally symmetric weight defined on R. Let rad(R) = $\mathfrak{m}$ = Rm = mR for some m in $\mathfrak{m} \setminus \mathfrak{m}^2$ and let e be the smallest positive integer such that $\mathfrak{m}^e = 0$. Then, R has the extension property with respect to the weight w if and only if $w(m^{e-1}) \neq 0$.

In [16], the author was able to prove a condition on general weight functions defined on finite chain rings. The result is more general than the previous theorem as it is for a weight that does not necessarily satisfy the condition that "all automorphisms of R preserve w and that all units u of R satisfy w(ur) = w(r) for every r in R". However, the condition is quite complex.

Codes over Finite Modules

In 1999, a group of mathematicians [10] developed a theory of codes taking their alphabet from modules that are defined over finite commutative rings instead of the alphabet being taken from finite fields or finite rings. Later in [6], this idea was generalized so that codes could take their alphabet from any finite module (defined over any arbitrary ring). Dinh and López-Permouth [3] attempted to characterize the finite rings and modules which satisfy the ring and/or module version of the MacWilliams Extension Theorem. They were able to prove some results for special cases of rings and modules. Greferath, Nechaev and Wisbauer had already shown in [6] that the extension theorem holds with respect to Hamming weight if the module we take our alphabet from is a Frobenius bi-module. And so once again, the question remained whether the full converse of Wood's extension theorem is true for Frobenius modules. Dinh and López [2], [3] were, however, able to lay out a strategy to prove the converse of the extension theorem. Their strategy was to reduce the proof of the converse of the extension problem to solving a problem concerning matrix modules. Following this strategy, Wood was able to prove the full converse of the extension theorem in 2006 [17]. We give a few definitions before stating known results for the extension theorem over finite modules.

***Definition 4.1* (Linear Code over a Module)**

Let R be a finite ring with unity and let A be a finite R-module, A will serve as the alphabet for the code. A linear code of length n over the alphabet A is a left R-submodule C of A^n.

In order to translate the MacWilliams extension theorem to the case of codes over modules, we still need a few more definitions. Also note that by convention, inputs to the homomorphisms of left R-modules will be written on the left of the homomorphism.

Definition 4.2 **(Monomial Transformation)**

A monomial transformation of A^n is an R-linear automorphism T of A^n of the form $(a_1,\dots,a_n)T = (a_{\sigma(1)}\tau_1,\dots,a_{\sigma(n)}\tau_n)$, where $(a_1,...,a_n)$ in A^n, σ is a permutation of {1,2,...,n} and $\tau_1,...,\tau_n$ in Aut(A). If $\tau_1,...,\tau_n$ in a subgroup G of Aut(A), we say that T is a G-monomial transformation of A^n.

Definition 4.3 **(Weight)**

Similar to weights defined over finite fields, a weight w on an R-module A is a function $w:A\to\mathbb{Q}$ with w(0)=0. This definition is naturally extended to A^n by defining $w(a_1,\dots,a_n) = \sum_{i=1}^{n} w(a_i)$.

Definition 4.4 **(Symmetry Groups)**

Given a weight $w:A\to\mathbb{Q}$, define the left and right symmetry groups of w by
$G_l := \{ u \in U(R) : w(ua)=w(a), \forall a \in A\}$.
$G_r := \{ \tau \in Aut(A) : w(a\tau)=w(a), \forall a \in A \}$.

In the above definition, U(R) denotes the group of units of the ring R. Note that G_r is the group of all automorphisms of A that preserve w. If a certain weight function w satisfies G_l= U(R) and G_r=Aut(A), we say that w is *maximally symmetric*.

Note that every τ in G_r preserves the weight w, consequently, a G_r-monomial transformation preserves w. Now we give the corresponding definition for monomial equivalence of codes over modules.

Definition 4.5 **(Monomial Equivalence)**

Assume the alphabet is the left R-module A, and that a weight w is defined on A with symmetry groups G_r and G_l. Let C_1, C_2 be two linear codes of length n over A. If there exists a G_r-monomial transformation T of A^n with C_1 T=C_2, we say that C_1 and C_2 are G_r-monomially equivalent.

Observe that if two codes are monomially equivalent via the transformation T, then the restriction of T to the code C_1 is an R-linear isomorphism that preserves the weight w. This proves one implication of the analogue of MacWilliams extension theorem when translated to codes over modules. The converse is described as a property, namely the "extension property".

***Definition 4.6* (Extension Property)**

The alphabet A has the extension property with respect to the weight w if: for any two linear codes C_1, C_2 of length n over A, if there is an R-linear isomorphism $f:C_1 \rightarrow C_2$ that preserves w, then f can be extended to a G_r-monomial transformation T of A^n.

We now wish to find necessary and sufficient conditions for an alphabet A equipped with a weight w to have the extension property. We first consider the Hamming weight, wt, defined, as for finite fields, by wt(0)=0 and wt(a)=1for nonzero a in A. Note that the symmetry groups are maximal for Hamming weight. First we define Frobenius bimodules.

***Definition 4.7* (Frobenius Bimodule)**

Let A be an (R, R) bimodule over the ring R. We say that A is Frobenius if ${}_RA \cong {}_R\hat{R}$ and $A_R \cong \hat{R}_R$, where $\hat{R}$ is the character ring of R.

Let R be a ring. If we consider our alphabet to be the ring R as a bimodule over itself, then we know by [6] that the extension property holds for R with respect to Hamming weight if R is Frobenius. This result is a consequence of the following theorem and the results known for codes over finite rings.

Theorem 4.1

A finite ring R is Frobenius if and only if ${}_RR_R$ is a Frobenius bimodule.

As a direct consequence of Theorems 4.1 and 3.1, we obtain the following result.

Theorem 4.2

If R is a finite Frobenius ring, then the alphabet A=R has the extension property with respect to Hamming weight.

The following theorem first appeared in [6], and in [19], Wood gives a proof that relies on generating characters and the linear independence of characters.

Theorem 4.3

Let R be a finite ring and A be a Frobenius bimodule over R. Then A has the extension property with respect to Hamming weight.

In [2], Dinh and López-Permouth were able to prove that the concepts of pseudo-injectivity and having the extension property are equivalent for codes of length 1. We give the definition of pseudo-injective followed by the proposition proved in [2].

***Definition 4.8* (Pseudo-Injective)**

A left module M over a ring R is pseudo-injective if, for every left R-submodule B of M and every injective R-linear mapping f: B →M, the mapping f extends to an R-linear mapping f': M → M.

Proposition 4.4

The alphabet A has the extension property for linear codes of length 1 with respect to Hamming weight if and only if A is a pseudo-injective module over R.

The previous proposition is used to prove a stronger result that was published in 2009 in [19] giving sufficient conditions for a module alphabet to satisfy the extension property.

Theorem 4.5

An alphabet A has the extension property with respect to Hamming weight if

i. A is pseudo-injective, and
ii. soc(A) is cyclic.

This ends our discussion of sufficient conditions and begins the search for necessary conditions, the elusive converse of the extension theorem for codes over finite rings or modules. As mentioned before, Wood was able to prove the full converse of the extension theorem by following the strategy laid down by Dinh and López-Permouth in [2] and [3]. Their strategy consisted of 3 steps (as explained by Wood in [17]):

1. If a finite ring R is not Frobenius, show that its socle contains a copy of a particular type of module defined over a matrix ring.
2. Show that counterexamples to the extension theorem exist in the context of linear codes defined over this particular matrix module.
3. Show that the counterexamples over the matrix module pull back to give counterexamples over the original ring.

Steps 1 and 3 were completed in [3] and Wood provided the counterexample required in step 2 completing the proof of the full converse in [17]. We state the main theorems for the three steps as given in [17].

***Theorem 4.6* (Step 1)**

If a finite ring R is not Frobenius, then there exists an i, with $1 \leq i \leq n$ and $k > \mu_i$ such that kT_i occurs in the direct sum decomposition of $soc({}_R R)$.

In the preceding theorem the μ_i's refer to the following isomorphism, $R/_{\mathrm{rad}(R)} \cong \mathbb{M}_{\mu_1}(F_{q_1}) \oplus \dots \oplus \mathbb{M}_{\mu_n}(F_{q_n})$,
where q_i are prime powers and F_{qi} is the finite field with q_i elements. And the T_i's are the simple left $\mathbb{M}_{\mu i}(F_{qi})$-modules $T_i := \mathbb{M}_{mi\times 1}(F_{qi})$.

***Theorem 4.7* (Step 2)**

Let R= $\mathbb{M}_m(F_q)$ be the ring of all m × m matrices over the finite field F_q, and let A=$\mathbb{M}_{m,k}(F_q)$ be the left R-module of all m × k matrices over F_q. If k>m, then there exist linear codes C, C' of length N over A, with $N = \prod(1+q_i)$, where the product is over i from 1 to k-1, and an R-linear isomorphism f: C → C'$ that preserves Hamming weight, yet C and C' are not monomially equivalent because one of the codes has an identically zero component, while the other does not. I.e. the alphabet A does not have the extension property.

***Theorem 4.8* (Step 3)**

Every finite ring that has the extension property with respect to Hamming weight is Frobenius.

This gives us the full converse of Wood's extension theorem if the alphabet in question is a finite ring. The following theorem [19] gives the converse of Theorem 4.5 and necessary conditions for a module to have the extension property.

Theorem 4.9

If the alphabet A has the extension property with respect to Hamming weight, then:

i. A is pseudo-injective, and
ii. soc(A) is cyclic.

Current Research

The answer to the main question regarding the extension theorem has been found for Hamming weight, however, the search continues for linear codes defined over modules equipped with arbitrary weight functions. In 2013, results were published in [5] putting sufficient conditions on weights to satisfy the extension theorem. However, these results apply to a special case of rings, namely, the direct product of finite chain rings, and to weights with maximal symmetry.

There are other areas of research; some mathematicians are considering particular rings and examining whether the rings satisfy MacWilliams Extension Theorem and MacWilliams identities with respect to certain weights. For example, the authors of [20] proved the MacWilliams identities for the complete, symmetrized and Lee weight enumerators for codes over the ring $\mathbb{Z}_4 + u\,\mathbb{Z}_4$, and they constructed self dual codes over this ring. Others have considered the ring $R_k=F_2[x_1,\ldots,x_k]/<x_1^2,\ldots,x_k^2>$ in [12], where it is proved that Lee, symmetric and Hamming weight enumerators are satisfied for codes over the ring R_k.

References

1. K. Bogart, D. Goldberg, and J. Gordon, *An Elementary Proof of the MacWilliams Theorem on Equivalence of Codes*, Inform. and Control, Vol. 37, 1978, p. 19-22.
2. H. Q. Dinh, and S. R. López-Permouth, *On the Equivalence of Codes over Finite Rings*, Appl. Algebra Engrg. Comm. Comput., Vol. 15, no.1, 2004, p. 37-50.
3. H. Q. Dinh, and S. R. López-Permouth, *On the Equivalence of Codes over Rings and Modules*, Finite Fields Appl., Vol. 10, no.4, 2004, p. 615-625.
4. M. Greferath, *Orthogonality Matrices for Modules over Finite Frobenius Rings and MacWilliams' Equivalence Theorem*, Finite Fields Appl., Vol. 8, no. 3, 2002, p. 323-331.
5. M. Greferath, C. Mc Fadden, and J. Zumbrägel, *Characteristics of Invariant Weights Related to Code Equivalence over Rings*, Designs, Codes and Cryptography Vol. 66 (2013), no. 1-3, p. 145-156.
6. M. Greferath, A. Nechaev, and R. Wisbauer, *Finite Quasi-Frobenius Modules and Linear Codes*, J. Algebra Appl., Vol. 3, no. 3, 2004, p. 247-272.
7. M. Greferath, and S. E. Schmidt, *Finite Ring Combinatorics and MacWilliams Equivalence Theorem*, J. Combin. Theory, Ser. A, Vol. 92, 2000, p. 17-28.
8. R. Hammons, Jr., P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Solé, *The $\mathbb{Z}_4$ Linearity of Kerdock, Preparata, Goethals, and Related Codes*, IEEE Trans. Inform. Theory, Vol. 40, 1994, p. 301-319.
9. W. C. Huffman, and V. Pless, *Fundamentals of Error-Correcting Codes*, Cambridge University Press, New York, 2003.
10. V. L. Kurakin, A. S. Kuzmin, V. T. Markov, A. V. Mikhalev, and A. Nechaev, *Linear Codes and Polylinear Recurrences over Finite Rings and Modules*, Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, Honolulu, HI, 1999, Proceedings of the 13th International Symposium AAECC-13, Lecture Notes in Computer Science, Vol. 1719, Springer, Berlin, 1999, p. 365-391.
11. F. J. MacWilliams, *Combinatorial Properties of Elementary Abelian Groups*, Ph.D. dissertation, Harvard Univerdity, Cambridge, MA, 1962.

12. H. Tapia-Recillas, and A. Torres-Chàzaro, *MacWilliams Identities over* R_k, Int. J. of Algebra, Vol. 7, no. 7, 2013, p. 341-348.
13. H. N. Ward, and J. A. Wood, *Characters and the Equivalence of Codes*, J. Combin. Theory Ser. A, Vol. 73, 1996, p. 348-352.
14. J. A. Wood, *Duality for Modules over Finite Rings and Applications to Coding Theory*, Amer. J. of Math., Vol. 121, 1999, p. 555-575.
15. J. A. Wood, Extension Theorems for Linear Codes over Finite Rings, "Applied Algebra, Algebraic Algorithms and Error-Correcting Codes" (T. Mora and H. Mattson), Springer-Verlag, New York/Berlin, 1997, p. 320-340.
16. J. A. Wood, *Weight Functions and the Extension Theorem for Linear Codes over Finite Rings*, Finite Fields: Theory, Applications, and Algorithms (R. C. Mullin and G. L. Mullen), Waterloo, ON, 1997.
17. J. A. Wood, *A Coding Theoretic Characterization of Finite Frobenius Rings*, manuscript, February 2006.
18. J. A. Wood, *Code Equivalence Characterizes Finite Frobenius Rings*, Proc. Amer. Math. Soc., Vol. 136, 2008, p. 699-706.
19. J. A. Wood, *Foundations of Linear Codes Defined over Finite Modules: The Extension Theorem and MacWilliams Identities*, Codes over Rings, Proceedings of the CIMPA Summer School, Ankara, Turkey, 18-29 August 2008, (Patrick Solé) Series on Coding Theory and Crytology, Vol. 6, World Scientific, Singapore, 2009, p. 124-190.
20. B. Yildiz, and S. Karadeniz, *Linear Codes over* $\mathbb{Z}_4 + u\,\mathbb{Z}_4$*: MacWilliams Identities, Projections, and Formally Self-Dual Codes,* partially presented in the 13th International Workshop on Algebraic and combinatorial coding theory, Pomorie, Bulgaria, 2012, preprint, July 2013.