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Exact Traveling Wave Solutions for the BBM Equation, Schamel Equation and Modified Kawahara Equation and their Geometric Interpretations

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Abstract: *In this paper, we employ the extended mapping method to obtain the exact traveling wavesolutions of the Benjamin-Bona-Mahony (BBM) equation, the Schamel equation and the modified Kawahara equation. Our results show that these solutionsinclude periodic wavesolutions and solitary wave solutions.The geometric interpretation for some of these solstionare introduced. The solitary wave solutions are obtained as a limiting case.*

Keywords: *Traveling wave solutions, Benjamin-Bona-Mahony (BBM) equation, the Schamelequation, the modified Kawahara equation, mapping method, geometric interpretations.*

Introduction

Many phenomena in physics and other field are often described by non linear partial differential quations (NLPDEs) particularly in fluidmecha-nics, solid state physics, plasmaphysics, and non-linear optics.The investigation of exact solutions of NLPDEs will help one to understand these phenomena better. There are many methods that have been used to construct exact traveling wave solutions for NLPDEs in the past decades, such as the inversescattering method [1], thetanh-function method [2], the extendedtanh-function method [3], Kudryashov method [4], the first integral method [5], and the homogeneous balance method [6]. Recently, some methods were presented to constructexactsolutions expressedinter-msof Jacobi elliptic functions (JEFs) fornonlinear evolution equations (NLEEs). Among them the Jacobi elliptic function expansion method [7, 8], the F-expansion method [9, 10], the generalized Jacobi elliptic function method [11], mapping method [12], extended mapping method [13-15] and other methods [16-20]. Actually, the Jacobi elliptic function method is just a special case of the mapping method under certain conditions. We demonstrate applications of the extended mapping method for finding exact solutions of three nonlinear evolution equations. The first of these equations is the BBM equation $u_t + \alpha u_x + u u_x + \beta u_{xxt} = 0$ the second equation is the schamel equation $u_t + \alpha u^{\frac{1}{2}} u_x + \beta u_{xxx} = 0$ the last equation is the modified Kawahara equation $u_t + u^2 u_x + \alpha u_{xxx} + \beta u_{xxxxx} = 0$.

In this paper, we apply the extended mapping method to construct more general exact solutions of LPDEs and introduce the geometricinterpretation for some of these

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solutions. This work is organized as follows. In Sections 2 we give brief descriptions of the extended mapping method and the geometric interpretation. In Sections 3-5 we construct traveling wave solutions for the BBM equation, Schamel equation and modified Kawahara equation, respectively. In the last Section, we summarize and discuss our results.

Description of method

In this section, we briefly describe the extended mapping method [13-15]. The main steps are summarized as in the following. For a given NLPDE, say, in two independent variables

$$G(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0. \qquad (2.1)$$

In general, the left hand side of Eq. (2.1) is a polynomial in u and its various derivatives.

Step 1: We seek the traveling wave solution of (2.1) in the form

$$u(x,t) = u(\xi);\ \ \xi = \kappa(x - \omega t) + \xi_0 \qquad (2.2)$$

where $\kappa$and $\omega$are constants to be determined later and $\xi_0$is an arbitrary constant.ThenEq. (2.1) is transformed to the ordinary differential equation (ODE)

$$H(u, u', u'', \ldots) = 0, \qquad (2.3)$$

where $u' = \frac{du}{d\xi}$and H is a polynomial of u and its various derivatives.If H is not a polynomial of u and its various derivatives, then we may use new variables$v = v(\xi)$ which makes Hbecome a polynomial of $v$and its various derivatives

.

Step 2: We assume that the solutions of Eq.(2.3) can be expressed in the form

$$u(\xi) = a_0 + \sum_{i=1}^{N}(a_i\phi^i(\xi) + b_i\phi^{-i}(\xi)) + \sum_{i=2}^{N} c_i\phi^{i-2}(\xi)\phi'(\xi) + \sum_{i=-1}^{-N} d_i\phi^i(\xi)\phi'(\xi), \qquad (2.4)$$

where N in Eq.(2.4) is a positive integer that can be determined by balancing the nonlinearterm (s) with the highest derivative term in (2.3) and$a_0,\ a_i,\ b_i,\ c_i$ and $d_i$are constants to be determined. The function$\phi(\xi)$ satisfies the nonlinear ODE $(\phi'(\xi))^2 = q_0 + q_2\phi^2(\xi) + q_4\phi^4(\xi), \qquad (2.5)$ where q_0, q_2 and $q_4$are constants.

Step 3: Substituting (2.4) with (2.5) into the ODE (2.3) and setting each coefficients of $((\phi'(\xi))^j\phi^i(\xi),\ j = 0,1,\ i = 0, \pm 1, \pm 2, \ldots)$ to zero to drive a system of algebraic equations for$a_0,\ a_i,\ b_i,\ c_i,\ d_i,\ \kappa$ and ω. Solving the system for$a_0,\ a_i,\ b_i,\ c_i,\ d_i,\ \kappa$ and ω. With the aid of Maple or Mathematica. Substituting the obtained coefficients into(2.4), then concentrationformulas
of traveling wave solutions of the NLPDE(2.1) can be obtained.

Step 4: Select the values of q_0, q_2, q_4 and the corrponding JEFs$\phi(\xi)$from Appendix A and substitute them into the concentration formulas of solutions to obtain the explicit and exact JEF solutions of Eq. (2.1). Various solutions of Eq. (2.5) were constructed using JEFs(see Appendix A), and these results were exploited in the design of a procedure for generating solutions of NLPDEs. The JEFssnξ = sn(ξ, m), cnξ =cn(ξ, m), anddnξ = dn(ξ, m), where $(0 < m < 1)$ is the modulus of the elliptic function, are double periodic and posses the following properties:

$sn^2\xi + cn^2\xi=1, \quad dn^2\xi + m^2\, sn^2\xi =1.$

$$\frac{d}{d\xi}(sn\xi) = cn\xi\; dn\xi, \qquad \frac{d}{d\xi}(cn\xi) = -sn\xi\; dn\xi,$$
$$\frac{d}{d\xi}(dn\xi) = -m^2 sn\xi\; cn\xi.$$

In addition when m$\rightarrow 1$, the functionssnξ, cnξ, and dnξdegenerate astanhξ, sechξ and sechξ
respectively. Some more properties of JEFs can be found in [21].
In order to describe the geometric interpretation for the solution of (2.1), we write thesolution of (2.1) at the regular regions (there is no singularities) in the form

$$u = u(x,t), \qquad u \in C^2 \qquad (2.6)$$

which describes 2- dimensional surfacesin $\mathcal{R}^3$.
To do that let us introduce the associatedMonge
formula as follows: $M = (x,t,u(x,t))$, which enables us to compute the most important geometric quantities [22] such as the Gaussian curvature K and mean curvature H. We can find the Gaussian and mean curvaturethrough the following steps:

$$K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}, \quad H = \frac{L_{11}g_{22} + L_{22}g_{11} - 2L_{12}g_{11}}{2(g_{11}g_{22} - g_{12}^2)},$$

$$g_{11}g_{22} - g_{12}^2 \neq 0, \qquad (2.7)$$

where $g_{11} = M_x \cdot M_x, \;\; g_{12} = M_x \cdot M_t, \; g_{22} = M_t \cdot M_t,$

$$L_{11} = M_{xx} \cdot N, \qquad L_{12} = M_{xt} \cdot N, \qquad L_{22} = M_{tt} \cdot N$$

and $N = \frac{M_x \times M_t}{\|M_x \times M_t\|}$. Here $g_{11} > 0,\; g_{22} > 0$ are the squares of thespeeds of the x and t parameter curves of M andg_{12} measures the coordinate angleθbetweenM_x and M_t(the tangents to the coordinates curves).

JEF solutions of the BBM equation

In this section, we consider the BBM equation [23]

$$u_t + \alpha u_x + u u_x + \beta u_{xxt} = 0, \qquad (3.1)$$

where α *and* $\beta$are constants. We referred to this equation as the BBM equation. Which wasfirst introduced by Benjamin et al [23]. As an improvement of the Korteweg-de Vries (KdV) equation for modeling long waves of small amplitude in 1+1 dimensions. The BBM equation describes the uni-directional propagation of small-amplitude long waves on the surface ofwater in a channel. Fu et al [24] used the JEF method and Alofi [25] used extended Jacobielliptic function expansion method and obtained the periodic wave solutions of Eq. (3.1). Here we obtain severalclasses of exact solutions of BBM equation expressed by various JEFsby using theextended mapping method and the availability of symbolic computation. In order to obtain the exact solutions of Eq. (3.1), substituting (2.2) into (3.1), we have $(\alpha-\omega)u' + uu' - \beta\omega u''' = 0.$ (3.2)

The balancing procedure implies that N = 2. Therefore the solution of Eq. (3.2) takes the form $u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + \frac{b_1}{\phi(\xi)} + \frac{b_2}{\phi^2(\xi)} + c_2\phi'(\xi) + \frac{d_1}{\phi(\xi)} + \frac{d_2}{\phi^2(\xi)}.$ (3.3)
Substituting (3.3) into (3.2) we can drive a system of algebraic equations for$a_0,\ a_1,\ a_2,\ b_1,\ b_2,\ c_2,\ d_1,\ d_2,\ \kappa$ *and* ω.Solving the algebraic equations by use of Maple or Mathematica.Therefor we get the following concentration formulas of traveling wave solutions of the BBMequation (3.1):

$$u = \omega - \alpha + 4\omega\beta\kappa^2[q_2 + 3q_4\phi^2(\xi)], \qquad (3.4)$$

$$u = \omega - \alpha + 4\omega\beta\kappa^2\left[q_2 + \frac{3q_0}{\phi^2(\xi)}\right], \qquad (3.5)$$

$$u = \omega - \alpha + 4\omega\beta\kappa^2\left[q_2 + 3q_4\phi^2(\xi) + \frac{3q_0}{\phi^2(\xi)}\right], \qquad (3.6)$$

$$u = \omega - \alpha + \omega\beta\kappa^2[q_2 + 6q_4\phi^2(\xi) \pm 6\sqrt{q_4}\phi'(\xi)], \qquad (3.7)$$

$$u = \omega - \alpha + \omega\beta\kappa^2\left[q_2 + 6q_0\phi^2(\xi) \pm 6\sqrt{q_0}\frac{\phi'(\xi)}{\phi^2(\xi)}\right], \qquad (3.8)$$

$$u = \omega - \alpha + \omega\beta\kappa^2\left[q_2 - 6\sqrt{q_4q_0} + 6q_4\phi^2(\xi) + \frac{6q_0}{\phi^2(\xi)} \pm 6\sqrt{q_4}\phi'(\xi) \pm 6\sqrt{q_0}\frac{\phi'(\xi)}{\phi^2(\xi)}\right], \qquad (3.9)$$

$$u = \omega - \alpha + \omega\beta\kappa^2\left[q_2 - 6\sqrt{q_4q_0} + 6q_4\phi^2(\xi) + \frac{6q_0}{\phi^2(\xi)} \mp 6\sqrt{q_4}\phi'(\xi) \pm 6\sqrt{q_0}\frac{\phi'(\xi)}{\phi^2(\xi)}\right]. \qquad (3.10)$$

With the aid of Appendix A and formula (3.4) and (3.5), one can get the periodic wavesolutions of Eq. (3.2) $u_1 = \omega - \alpha + 4\omega\beta\kappa^2[-1 - m^2 + 3m^2 sn^2\xi],$ (3.11)
we can also find some exact solution of(3.2) expressed by rational expressions of JEFs

$$u_{2,3} = \omega - \alpha + \omega\beta\kappa^2\left[2m^2 - 4 \quad + 3m^2\left(\frac{sn\xi}{1 \pm dn\xi}\right)^2\right], \qquad (3.12)$$

$$u_{4,5} = \omega - \alpha + \omega\beta\kappa^2\left[2m^2 + 1 \quad + 3(1 - m^2)^2\left(\frac{sn\xi}{cn\xi \pm dn\xi}\right)^2\right]. \qquad (3.13)$$

With the aid of Appendix A and the formulas (3.6) -(3.10), we obtain the following exactsolutions of (3.2):

$$u_6 = \omega - \alpha + \omega\beta\kappa^2 \left[2 - 4m^2 + (msn\xi + idn\xi)^2 + \frac{3}{(msn\xi + idn\xi)^2}\right], \qquad (3.14)$$

$$u_{7,8} = \omega - \alpha + \omega\beta\kappa^2[2m^2 - 1 - 6m^2 cn^2\xi \pm 6imsn\xi dn\xi], \qquad (3.15)$$

$$u_{9,10} = \omega - \alpha + \frac{\omega\beta\kappa^2}{2}[m^2 - 2 + 3m^2(sn\xi \pm icn\xi)^2 \pm 6mdn\xi(cn\xi \mp isn\xi)], (3.16)$$

$$u_{11,12} = \omega - \alpha + \omega\beta\kappa^2\left[2 - m^2 + 6\sqrt{1 - m^2} - 6dn^2\xi + 6(m^2 - 1)nd^2\xi \pm 6im^2sn\xi cn\xi \mp 6m^2\sqrt{m^2 - 1}sd\xi cd\xi\right]. \qquad (3.17)$$

Other JEFs are omitted here for simplicity. The periodic wave solution (3.11) was given in[24]. Compared with the results given in [24] we find more new solutions. As $m \to 1$, equations (3.11), (3.12) and reduce to the solitary wave solutions $u_{13}(x,t) = \omega - \alpha + 4\omega\beta\kappa^2[-2 + 3tanh^2(\kappa(x - \omega t) + \xi_0)]$, (3.18)

$$u_{14,15}(x,t) = \omega - \alpha + \omega\beta\kappa^2\left[-2 + 3\left(\frac{tanh(\kappa(x - \omega t) + \xi_0)}{1 \pm sech(\kappa(x - \omega t) + \xi_0)}\right)^2\right]. \quad (3.19)$$

The solutions (3.18) represent surfaces whose Gaussian curvature K and mean curvature H aregiven by $K = 0$, $H = \frac{12\omega(1+\omega^2)(1-3tanh^2\xi)sech^2\xi}{(1+576(1+\omega^2)\omega^2\beta^2\kappa^6 tanh^2\xi sech^4\xi)^{\frac{3}{2}}}$, $\xi = \kappa(x - \omega t) + \xi_0$.

Thus the solution (3.18) represents a family of parabolic surfaces$(K = 0, \ H \neq 0)$ and a family of planes $(K = 0, \ H = 0)$ on the points of the cuspidal edge $x = \omega t - \frac{\xi_0}{\kappa} + \frac{1}{\kappa}tanh^{-1}\left(\pm\frac{1}{\sqrt{3}}\right)$ as shown in Fig. 1. These planes of (3.18) are given by the vector equation$M = (x, t, \omega - \alpha - 4\omega\beta\kappa^2)$.The solution (3.18) have singularities at the points $x = \omega t - \frac{\xi_0}{\kappa}$.

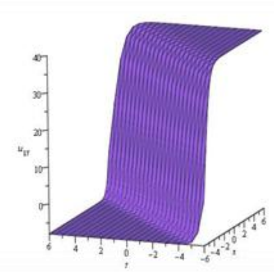

Figure 1: Graph ofu_{13}in (3.18) for $\alpha = \beta = \kappa = \omega = \xi_0 = 1$

JEF solutions of schamel equation

Let us consider the schamel equation [26]

$$u_t + \alpha u^{\frac{1}{2}} u_x + \beta u_{xxx} = 0 \qquad (4.1)$$

where α and β are constants.This equation describing ion-acoustic wave in a cold-ion plasma where electron do not behave isothermally during their passage of the wave. Schamel [26] derived this equation and a simple solitary wave solution having a sech^{4}profile was obtained.Therefore the Schamel equation (4.1) containing a square root nonlinearity is very attractivemodel for the study of ion-acoustic waves in plasmas and dusty plasmas. Khater et al [27] have obtained abundant exact solutions in terms of JEFs of the Schamel equation by meansof mapping method.

Some nonlinear models in plasma are described by canonical models including the KdV, modified KdV, Zakharov-Kuznetsov and the Kawahara equations. The KdV, the Schameland the Zakharov-Kuznetsov equations can be derived by many authors [16, 26, 28] in fluid dynamics and ion-acoustic wave in plasma. El-Kalaawy [29] studied the exact solitary wavesolutions of Schamel equation in plasma with negative ions. Hassan [30] obtained abundantnew exact of the Schamel-Korteweg-de Vries (S-KdV) equation and modified Zakharov-Kuznetsov (mZK) equation arising in plasma and dust plasma. In order to obtain the exact solutions of Eq. (4.1) we use the transformation $u(x,t) = v^2(x,t)$, $v(x,t) = V(\xi)$, $\xi = \kappa(x - \omega t) + \xi_0$, to reduce Eq.(4.1) to the ODE

$$-\omega V V' + \alpha V^2 V' + \beta \kappa^2 (V V''' + 3V' V'') = 0. \qquad (4.2)$$

The balancing procedure implies that N = 2. Therefore, the solution of Eq. (4.2) takes theform

$$V(\xi) = a_0 + a_1 \phi(\xi) + a_2 \phi^2(\xi) + \frac{b_1}{\phi(\xi)} + \frac{b_2}{\phi^2(\xi)} + \qquad c_2 \phi'(\xi) + \frac{d_1}{\phi(\xi)} + \frac{d_2}{\phi^2(\xi)}. \quad (4.3)$$

Substituting (4.3) into (4.2) we obtain a system of algebraic equations for $a_0, a_1, a_2, b_1, b_2, c_2, d_1, d_2, \kappa$ and ω.

Solving this system we get the following concentration formulas oftraveling wave solutions of the Schamel equation (4.1):

$$u = \frac{100 \beta^2 \kappa^4}{\alpha^2} \left[-q_2 \pm \sqrt{q_2^2 - 3 q_0 q_4} - 3 q_4 \phi^2(\xi) \right]^2, \qquad (4.4)$$

$$u=\frac{100\beta^2\kappa^4}{\alpha^2}\left[-q_2\pm\sqrt{q_2^2-3q_0q_4}-\frac{3q_0}{\phi^2(\xi)}\right]^2, \quad (4.5)$$

with $\xi=\kappa\left(x\mp 16\beta\kappa^2\sqrt{q_2^2-3q_0q_4}t\right)+\xi_0,$

$$u=\frac{100\beta^2\kappa^4}{\alpha^2}\left[-q_2\pm\sqrt{q_2^2+12q_0q_4}-3q_4\phi^2(\xi)-\frac{3q_0}{\phi^2(\xi)}\right]^2, \quad (4.6)$$

with $\xi=\kappa\left(x\mp 16\beta\kappa^2\sqrt{q_2^2+12q_0q_4}t\right)+\xi_0,$

$$u=\frac{25\beta^2\kappa^4}{4\alpha^2}\left[-q_2\pm\sqrt{q_2^2+12q_0q_4}-6q_4\phi^2(\xi)\pm 6\sqrt{q_4}\phi'(\xi)\right]^2, \quad (4.7)$$

$$u=\frac{25\beta^2\kappa^4}{4\alpha^2}\left[-q_2\pm\sqrt{q_2^2+12q_0q_4}-\frac{6q_0}{\phi^2(\xi)}\pm 6\sqrt{q_0}\frac{\phi'(\xi)}{\phi^2(\xi)}\right]^2, \quad (4.8)$$

with $\xi=\kappa\left(x\mp 4\beta\kappa^2\sqrt{q_2^2+12q_0q_4}t\right)+\xi_0,$

$$u=\frac{25\beta^2\kappa^4}{4\alpha^2}\Bigg[-q_2+6\sqrt{q_0q_4}\pm\sqrt{q_2^2+60q_2\sqrt{q_0q_4}+123q_0q_4}$$

$$-6q_4\phi^2(\xi)-\frac{6q_0}{\phi^2(\xi)}\pm 6\sqrt{q_4}\phi'(\xi)\pm 6\sqrt{q_0}\frac{\phi'(\xi)}{\phi^2(\xi)}\Bigg]^2,$$

(4.9)

with $\xi=\kappa\left(x\mp 4\beta\kappa^2\sqrt{q_2^2+60q_2\sqrt{q_0q_4}+123q_0q_4}t\right)+\xi_0,$

With the aid of Appendix A and formula (4.4), one can get the periodic wave solutions of Eq. (4.1):

$$u_{1,2}=\frac{100\beta^2\kappa^4}{\alpha^2}\left[1+m^2\pm\sqrt{m^4-m^2+1}-3m^2sn^2\xi\right]^2, \quad (4.10)$$

$\xi=\kappa\left(x\mp 16\beta\kappa^2\sqrt{m^4-m^2+1t}\right)+\xi_0,$

$$u_{3,4}=\frac{25\beta^2\kappa^4}{4\alpha^2}\left[-2(m^2+1)\pm\sqrt{m^4+14m^2+1}+3(mcn\xi\pm dn\xi)^2\right]^2, \quad (4.11)$$

$\xi=\kappa\left(x\mp 4\beta\kappa^2\sqrt{m^4+14m^2+1t}\right)+\xi_0,$

$$u_{5,6}=\frac{25\beta^2\kappa^4}{4\alpha^2}\left[-2(m^2-2)\pm\sqrt{m^4-16m^2+16}-3m^2(sn\xi\pm cn\xi)^2\right]^2, \quad (4.12)$$

$\xi=\kappa\left(x\mp 4\beta\kappa^2\sqrt{m^4-16m^2+16t}\right)+\xi_0.$

With the aid of Appendix A and the formulas (4.5)-(4.9), we can obtain more general types of exact solutions of (4.1):

$$u_{7,8} = \frac{100\beta^2\kappa^4}{\alpha^2}\left[m^2 - 2 \pm \sqrt{m^4 - 16m^2 + 16} + \qquad 3dn^2\xi + (1-m^2)nd^2\xi\right]^2, \quad (4.13)$$
$$\xi = \kappa\left(x \mp 16\beta\kappa^2\sqrt{m^4 - 16m^2 + 16}t\right) + \xi_0,$$

$$u_{9,10} = \frac{25\beta^2\kappa^4}{\alpha^2}\left[2 - m^2 \pm 2\sqrt{m^4 - m^2 + 1} - \frac{3m^4}{2}\left(\frac{sn\xi}{1 \pm dn\xi}\right)^2 - \frac{3}{2}\left(\frac{1 \pm dn\xi}{sn\xi}\right)^2\right]^2,$$
(4.14)

$$\xi = \kappa\left(x \mp 16\beta\kappa^2\sqrt{m^4 - m^2 + 1}t\right) + \xi_0,$$

$$u_{11,12} = \frac{25\beta^2\kappa^4}{4\alpha^2}\left[1 + m^2 \pm \sqrt{m^4 + 14m^2 + 1} - 6m^2sn^2\xi + 6mcn\xi dn\xi\right]^2,$$
(4.15)
$$\xi = \kappa\left(x \mp 4\beta\kappa^2\sqrt{m^4 + 14m^2 + 1}t\right) + \xi_0,$$

$$u_{13,14} = \frac{25\beta^2\kappa^4}{4\alpha^2}\left[m^2 - 2 \pm \sqrt{m^4 - 16m^2 + 16} + 6dn^2\xi \mp 6im^2sn\xi cn\xi\right]^2,$$
(4.16)
$$\xi = \kappa\left(x \mp 4\beta\kappa^2\sqrt{m^4 - 16m^2 + 16}t\right) + \xi_0.$$

$$u_{15,16} = \frac{25\beta^2\kappa^4}{4\alpha^2}[1 + 6m + m^2 \pm \sqrt{m^4 - 60m^2 - 134m^2 - 60m + 1} - 6m^2sn^2\xi - 6ns^2\xi \pm 6mcn\xi dn\xi \pm cs\xi ds\xi]^2, \quad (4.17)$$
$$\xi = \kappa\left(x \mp 4\beta\kappa^2\sqrt{m^4 - 60m^2 - 134m^2 - 60m + 1}t\right) + \xi_0$$

Other JEFs are omitted here for simplicity. The solutions (4.5) and (4.6) were given in[29] and (4.11) and (4.12) were given in [27]. As $m \to 1$, equations (4.11), (4.13) and reduce to the solitary wave solutions

$$u_{17} = \frac{900\beta^2\kappa^4}{\alpha^2}sech^4(\kappa x - 16\beta\kappa^3t + \xi_0),$$

$$u_{18} = \frac{100\beta^2\kappa^4}{\alpha^2}[3 - tanh^2(\kappa x + 16\beta\kappa^3t + \xi_0)]^2$$
(4.18)
$$u_{19,20} = \frac{25\beta^2\kappa^4}{4\alpha^2}[2 \pm 1 - 3(\tanh(\kappa x \mp 4\beta\kappa^3t + \xi_0) \pm isech(\kappa x \mp 4\beta\kappa^3t + \xi_0))^2]^2.$$
(4.19)

The solutions (4.18) represent surfaces whose Gaussian curvature K and mean curvature H $K_{1,2} = 0$, $H_1 = \frac{1800\alpha^4\beta^2\kappa^6(1 + 256\beta^2\kappa^4)(4cosh^2\xi - 5)sech^6\xi}{(\alpha^4 + 12960000\beta^4\kappa^{10}(1 + 256\beta^2\kappa^4)sech^8\xi tanh^2\xi)^{\frac{3}{2}}}$, $\xi = \kappa x - 16\beta\kappa^3t + \xi_0$

$$H_2 = \frac{-600\alpha^4\beta^2\kappa^8(1+256\beta^2\kappa^4)(4cosh^2\xi - 18cosh^2\xi + 15)sech^2\xi}{(\alpha^4 + 1440000\beta^4\kappa^{10}(1+256\beta^2\kappa^4)sech^4\xi tanh^2\xi(1-3tanh^2)^2)^{\frac{3}{2}}}$$

$\xi = \kappa x + 16\beta\kappa^3 t + \xi_0.$

Thus, the solutionsu_{17}, u_{18}represent a family of parabolic surfaces$(K = 0,\ \ H \neq 0)$ and a family of planes $(K = 0,\ H = 0)$ on the points of the cuspidal edge $x = 16\beta\kappa^2 t - \frac{\xi_0}{\kappa} + \frac{1}{\kappa}cosh^{-1}\left(\pm\frac{\sqrt{5}}{2}\right)$ and $x = -16\beta\kappa^2 t - \frac{\xi_0}{\kappa} + \frac{1}{\kappa}cosh^{-1}\left(\pm\frac{1}{2}\sqrt{9 \pm \sqrt{21}}\right)$ these planes of (4.18) aregiven by the vector equations $M = \left(x, t, \frac{576\beta^2\kappa^4}{\alpha^2}\right)$ and$M = \left(x, t, \frac{400\beta^2\kappa^4(3\pm\sqrt{21})^2}{(9\pm\sqrt{21})\alpha^2}\right)$.The solutions given by Eq. (4.18) have singularities at the points $x = \pm 16\beta\kappa^2 t - \frac{\xi_0}{\kappa}.$ The surfaces and their singularities are shown in Fig 2.

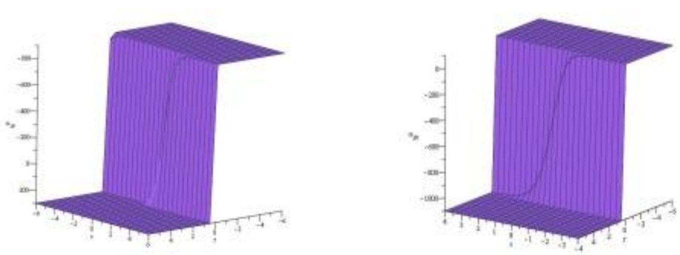

Figure 2: Graph ofu_{17} and u_{18}in (4.18) for $\alpha = \beta = \kappa = \omega = \xi_0 = 1$

JEF solutions of modi_ed Kawahara equation

The modified Kawahara equation is [31] $\boldsymbol{u_t + u^2 u_x + \alpha u_{xxx} + \beta u_{xxxxx} = 0},$ (5.1)

where α and β are constants. This equation occurs in the theory of magneto-acoustic wavesin plasmas and propagation of nonlinear water-waves in the long-wavelength region as in thecase of KdV's equations. Due to the wide range of applications of Eq. (5.1), it is importantto find exact solutions of the modified Kawahara equation. Traveling wave solutions ofmodified fifth order KdV equation and modified Kawahara equation have been studied in[32, 33]. Substituting (2.2) into (5.1) we have $-\omega u' + u^2 u' + \alpha\kappa^2 u''' + \beta\kappa^4 u''''' = 0.$ (5.2) The balancing procedure implies that N = 2. Therefore, we apply the extended mappingmethod with the ansatz solution (3.3) to obtain the solutions of Eq.(5.1). Substituting (3.3) into (5.2)we obtain a system of algebraic equations for $a_0,\ a_1,\ a_2,\ b_1, b_2,\ c_2, d_1, d_2, \kappa$ and $\omega.$Solving this system we get the following traveling wave solutions of the modified Kawahara equation (5.1):

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$$u = \mp \frac{\alpha + 20\beta\kappa^2 q_2}{\sqrt{-10\beta}} \pm 6\sqrt{-10\beta}\kappa^2 q_4 \phi^2(\xi), \quad (5.3)$$

$$u = \mp \frac{\alpha + 20\beta\kappa^2 q_2}{\sqrt{-10\beta}} \pm \frac{6\sqrt{-10\beta}\kappa^2 q_0}{\phi^2(\xi)}, \quad (5.4)$$

$$\xi = \kappa\left(x - \frac{-\alpha^2 - 240\beta^2\kappa^4 q_2^2 + 720\beta^2\kappa^4 q_0 q_4}{10\beta} t\right) + \xi_0,$$

$$u = \mp \frac{\alpha + 20\beta\kappa^2 q_2}{\sqrt{-10\beta}} \pm 6\sqrt{-10\beta}\kappa^2 q_4 \phi^2(\xi) \pm \frac{6\sqrt{-10\beta}\kappa^2 q_0}{\phi^2(\xi)}, \quad (5.5)$$

$$\xi = \kappa\left(x + \frac{\alpha^2 + 240\beta^2\kappa^4 q_2^2 + 2880\beta^2\kappa^4 q_0 q_4}{10\beta} t\right) + \xi_0,$$

$$u = \mp \frac{\alpha + 5\beta\kappa^2 q_2}{\sqrt{-10\beta}} \pm 3\sqrt{-10\beta}\kappa^2 q_4 \phi^2(\xi) \pm 3\sqrt{-10\beta q_4}\kappa^2 \phi'(\xi), \quad (5.6)$$

$$u = \mp \frac{\alpha + 5\beta\kappa^2 q_2}{\sqrt{-10\beta}} \pm \frac{3\sqrt{-10\beta}\kappa^2 q_0}{\phi^2(\xi)} \pm 3\sqrt{-10\beta q_0}\frac{\phi'(\xi)}{\phi^2(\xi)}, \quad (5.7)$$

$$\xi = \kappa\left(x + \frac{\alpha^2 + 15\beta^2\kappa^4 q_2^2 + 180\beta^2\kappa^4 q_0 q_4}{10\beta} t\right) + \xi_0.$$

From Appendix A and the formulas (5.3), we obtain the exact traveling wave solutions ofEq. (5.1)

$$u_{1,2} = \mp \frac{\alpha - 20\beta\kappa^2(1+m^2)}{\sqrt{-10\beta}} \pm 6\sqrt{-10\beta}\kappa^2 m^2 sn^2\xi,$$

(5.8)

$$\xi = \kappa\left(x - \frac{-\alpha^2 - 240\beta^2\kappa^4(1+m^2)^2 + 720\beta^2\kappa^4 m^2}{10\beta} t\right) + \xi_0,$$

$$u_{3,4} = \mp \frac{\alpha + 10\beta\kappa^2(1-2m^2)}{\sqrt{-10\beta}} \pm \frac{3}{2}\sqrt{-10\beta}\kappa^2\left(\frac{sn\xi}{1 \pm cn\xi}\right)^2, \quad (5.9)$$

$$\xi = \kappa\left(x - \frac{-\alpha^2 - 60\beta^2\kappa^4(1-2m^2)^2 + 45\beta^2\kappa^4}{10\beta} t\right) + \xi_0,$$

From Appendix A and the formulas (5.4), (5.5) and (5.6), we can obtain new and moregeneral types of exact solutions of (5.1)

$$u_{5,6} = \mp \frac{\alpha + 20\beta\kappa^2(2-m^2)}{\sqrt{-10\beta}} \pm 6\sqrt{-10\beta}\kappa^2 cs^2\xi$$

$$\pm 6\sqrt{-10\beta}\kappa^2(1-m^2)sc^2\xi, \qquad (5.10)$$

$$\xi = \kappa\left(x + \frac{\alpha^2 + 240\beta^2\kappa^4(2-m^2)^2 + 2880\beta^2\kappa^4(1-m^2)}{10\beta}t\right) + \xi_0,$$

$$u_{7,8} = \mp\frac{\alpha + 10\beta\kappa^2(m^2-2)}{\sqrt{-10\beta}} \pm \frac{3}{2}\sqrt{-10\beta}\kappa^2 m^2(sn\xi \pm icn\xi)^2 \pm \frac{3\sqrt{-10\beta}\kappa^2 m^2}{2(sn\xi \pm icn\xi)^2}, \qquad (5.11)$$

$$\xi = \kappa\left(x + \frac{\alpha^2 + 60\beta^2\kappa^4(m^2-2)^2 + 180\beta^2\kappa^4 m^4}{10\beta}t\right) + \xi_0,$$

$$u_{9,10} = \mp\frac{\alpha + 5\beta\kappa^2(2-m^2)}{\sqrt{-10\beta}} \mp 3\sqrt{-10\beta}\kappa^2 dn^2\xi \mp 3\sqrt{-10\beta}\kappa^2 m^2 sn\xi cn\xi. \qquad (5.9)$$

$$\xi = \kappa\left(x + \frac{\alpha^2 + 15\beta^2\kappa^4(2-m^2)^2 + 180\beta^2\kappa^4(1-m^2)}{10\beta}t\right) + \xi_0,$$

Other JEFs are omitted here for simplicity. The periodic wave solutions (5.8) were given [32] and [33]. In this paper we find many types of traveling wave solutions to Eq. (5.1). When $m \to 1$, equations (5.8) and (5.9) reduce to

$$u_{11,12} = \mp\frac{\alpha - 40\beta\kappa^2}{\sqrt{-10\beta}} \pm 6\sqrt{-10\beta}\kappa^2 tanh^2\xi, (5.10)$$

$$\xi = \kappa\left(x + \frac{\alpha^2 + 40\beta\kappa^2}{10\beta}t\right) + \xi_0,$$

$$u_{13,14} = \mp\frac{\alpha - 10\beta\kappa^2}{\sqrt{-10\beta}} \pm \frac{3}{2}\sqrt{-10\beta}\kappa^2\left(\frac{tanh\xi}{1 \pm sech\xi}\right)^2, (5.11)$$

$$\xi = \kappa\left(x + \frac{\alpha^2 + 15\beta\kappa^2}{10\beta}t\right) + \xi_0.$$

The solutions (5.10) represent surfaces whose Gaussian curvature K and mean curvature Hare given by $K_{1,2} = 0,$ $H_{1,2} = \pm\frac{\sqrt{-10\beta}\frac{3\kappa^4}{50\beta^2}(100\beta^2 + (\alpha^2 + 240\beta^2\kappa^4)^2)(1 - 3tanh^2\xi)sech^2\xi}{\left(1 - \frac{72\kappa^6}{5\beta}(100\beta^2 + (\alpha^2 + 240\beta^2\kappa^4)^2)sech^4\xi tanh^2\xi\right)^{\frac{3}{2}}}.$

Thus the solutions (5.10) represent a family of parabolicsurfaces $(K = 0, \ H \neq 0)$ and a family of planes$(K = 0, \ H = 0)$on the points of the cuspidal edge
$x = -\frac{\alpha^2 240\beta^2\kappa^4}{10\beta}t - \frac{\xi_0}{\kappa} + \frac{1}{\kappa}tanh^{-1}(\pm\frac{1}{\sqrt{3}})$

these planes of (5.10) are given by the vector equations $M = \left(x, t, \mp\frac{\alpha - 20\beta\kappa^2}{\sqrt{-10\beta}}\right)$.The solutions (5.1) have singularities at thepoints$x = -\frac{\alpha^2 240\beta^2\kappa^4}{10\beta}t - \frac{\xi_0}{\kappa}$.

The surfaces and their singularities are shown in Fig 3.

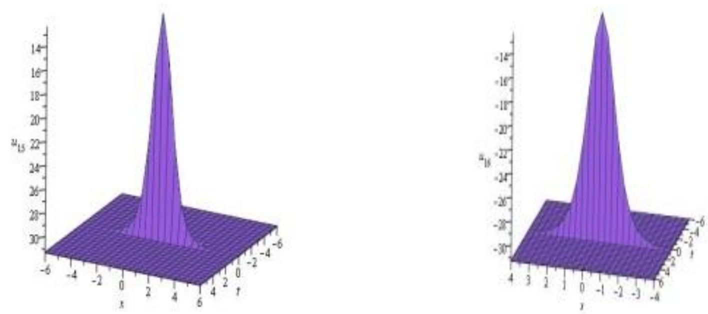

Figure 3: Graph ofu_{11} and u_{12}in (5.10) for $\alpha = \kappa = 1, \beta = -1$ and $\xi_0 = 0$.

Conclusions

In this paper, the extended mapping method has been applied to obtain many types of exact traveling wave solutions for the BBM equation, the Schamel equation and the modifiedKawahara equation. These solutions expressed by JEFs and hyperbolic functions. It shouldbe noted that, although many exact solutions are obtained in this work, it has beenshownthat some of these solution are the same as the results given in [24], [27], [29], [32] and[33]. The computer symbolic system such as Maple or Mathematica allow us to performcomplicated and tedious calculations. Moreover, the solitary wave solutions have been obtainedas a limiting case. Also, we discussed the geometric interpretations for some of thesesolutions. Geometrically the solutions given in this paper represented by parabolic surfaces $(K = 0,\ \ H \neq 0)$ and in some special cases family of planes $(K = 0,\ \ H = 0)$. Some of theconsidered surfaces may be contain singularities.

References

1. M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge, Cambridge University Press (1991).
2. W. Maliet, Solitary wave solutions of nonlinear wave equations, Am. J. Phys. 60 (1992) 650-654.
3. E.G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000) 212-218.

4. N. A. Kudryashov, One method for finding exact solutions of nonlinear differential equations , Comm. Non. Sci. Num. Simu17 (2012) 2248-2253.
5. N.Taghizadeh, M. Mirzazadeh and A. SamieiPaghaleh, The first integral method to nonlinear partial differential equations, Applic. Appl. Math 7 (2012) 117-132.
6. M. Wang, Solitary wave solutions for variant Boussinesq equation, Phys. Lett. A 199 (1995) 169-172.
7. S.K. Liu, Z.T. Fu, S.D. Liu and Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289 (2001) 69-74.
8. E.J. Parkes, B.R. Duffy and P.C. Abbott, The Jacobi elliptic-function method for finding periodic wave solutions to nonlinear evolution equations, Phys. Lett. A 295 (2002) 280-286.
9. Y.B. Zhou, M.L. Wang and Y.M. Wang, Periodic wave solutions to a coupled KdV equations with variable coe_cients, Phys. Lett. A 308 (2003) 31-36.
10. Liu and K. Yang, The extended F-expansion method and exact solutions of nonlinear PDEs, Chaos, Solitons and Fractals 22 (2004) 111-121.
11. H. T. Chen and H. Q. Zhang, New double periodic and multiple soliton solutions of the generalized (2+1)-dimensional Boussinesq equation, Chaos, Solitons and Fractals 20 (2004) 765-769.
12. Y.Z. Peng, A mapping method for obtaining exact travelling wave solutions to nonlinear evolution equations, Chin. J. Phys. 41 (2003) 103-110.
13. W.L. Zhang, G.J.Wu, M. Zhang, J.M.Wang and J.H. Han, New exact periodic solutions to (2+1)-dimensional dispersive long wave equations Chinese Phys. B 17 (2008) 1156-1164.
14. G.J. Wu, J. H. Han, W.L. Zhang and M. Zhang, New periodic wave solutions to non linear evolution equations by the extended mapping method, Physica D 229 (2007) 116-122.
15. M. A. Abdou and S. Zhang, New periodic wave solutions via extended mapping method, Commun. Nonlinear Sci. Numer. Simul. 16 (2009) 2-11.
16. P.G. Drazin and R.S. Johnson, Solitons: An Introduction, (Cambridge University Press, Cambridge 1989).
17. H. Khater and M. M. Hassan, Travelling and periodic wave solutions of some non linear wave equations, Z. Naturforsch. 59a (2004) 389-396.
18. H.T. Chen and H.Q. Zhang, Improved Jacobin elliptic function method and its applications, Chaos, Solitons and Fractals 15 (2003) 585-591.
19. M. A. Abdou, Further improved F-expansion and new exact solutions for nonlinear evolution equation, Nonlinear dyn. 52 (2008) 277- 288.
20. M. A. Abedel-Razek , A.K. Seddeek, N.H. Abdel-All , New exact Jacoby elliptic function solutions for nonlinear equtions using F-expansion method, Stud. Math. scien2 (2011) 88.
21. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, (Dover, New York 1965).
22. M. P. Do-Carmo , Di_erential geometry of curves and surfaces, Prentice-Hall Inc., New Jersey , (1976).

23. T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, Philos Trans R Soc London, Ser A272 (1972) 47-78.
24. Z.T.Fu, S.K.Lie, S.D.Lie and Q.Zhao, The JEFE method and periodic solutions of two kinds of nonlinear wave equtions, Commun. Nonlinear Sci. Numer. Simul. 8 (2003) 67-75.
25. S. Alofi, Extended Jacobi Elliptic Function Expansion Method for Nonlinear Benjamin-Bona-Mahony Equations, Intern. Math. Forum. 7 (2012) 2639 - 2649.
26. H. Schamel, A modi_edKortewegde-Vries equation for ion acoustic waves due to resonant electrons, J. Plasma Phys. 9 (1973) 377-387.
27. H. Khater, M. M. Hassan, E. V. Krishnan and Y. Z. Peng, Applications of elliptic functions to ion-acoustic plasma waves, Eur. Phys. J. D 50 (2008) 177-184.
28. V. E. Zakharov and E. A. Kuznetsov,On three-dimensional solitons, Sov. Phys. JETP39 (1974) 285- 288.
29. O. El-Kalaawy, Exact solitary solution of Schamel equation in plasmas with negative ions, Phys. Plasmas 18 (2011) 112302.
30. M. M. Hassan, New exact solutions of two nonlinear physical models, Com mun.Theor.Phys., 53 (2010) 596-604.
31. T. Kawahara, Oscillatory solitary waves in dispersive media, J. Phys. Soc. Jpn. 33 (1972) 260-264.
32. E. Yusufoglu and A. Bekir, Symbolic computation and new families of exact travelling solutions for the Kawahara and modified Kawahara equations, Comput. Math. Appl. 55 (2008)1113-1121.
33. H. Khater, M. M. Hassan and D. K. Callebaut, Traveling wave solutions to some important equations of mathematical physics, Reports on Math. Phys. 66 (2010) 1-19.