The Power of the Depth of Iteration in Defining Relations by Induction

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Abstract: *In this paper we provoke the question of whether the sequence* $IND[n] \subseteq IND[n^2] \subseteq IND[n^3] \subseteq \ldots$ *is strictly increasing, i.e., the question of whether increasing the depth of iteration increases the expressive power of defining by induction. Solving this question should have a deep impact on computer science as well as on mathematical logic since it is a question in a subject on the crossroads between them, namely, descriptive complexity. We shall mention a potential way of tackling the problem.*

Introduction

In 1979 Aho and Ullman noted that first-order logic is unable to express the transitive closure of a given relation, and suggested extending it by adding the least fixed-point operator [1],[2]: If $\varphi(R, x_1,..., x_k)$ is an R-positive first-order formula, where R is a relation symbol of arity k, then $(LFP_{R,x_1,...,x_k}\varphi)$ is interpreted in any finite structure A as the least fixed point of the map φ^{A} from k-ary relations on the universe of A to k-ary relations on the universe of A given by $\varphi^{\mathsf{A}}(S) = \{< a_1, \ldots, a_k > \in A^k : \mathsf{A} \vDash \varphi(S, a_1, \ldots, a_k)\}$.

Since φ is R-positive i.e. any occurrence of R in φ lies in the scope of an even number of negations, then the map φ^{A} is monotone, and hence $\varnothing \subseteq \varphi^{\mathsf{A}}(\varnothing) \subseteq (\varphi^{\mathsf{A}})^2(\varnothing) \subseteq (\varphi^{\mathsf{A}})^3(\varnothing) \subseteq \cdots$ and since A is finite, then there is $r \leq ||\mathsf{A}||^k$ such that $(\varphi^{\mathsf{A}})^r(\varnothing) = (\varphi^{\mathsf{A}})^{r+1}(\varnothing)$. It can be easily seen that $(\varphi^{\mathsf{A}})^r(\varnothing)$ is the least fixed point. [3]

Example

In finite graphs, the reflexive transitive closure of the edge relation is the least fixed point of the formula $\varphi(R, x, y) =: x = y \vee \exists z(E(x,z) \wedge R(z, y))$ i.e. for any u, v there is a path from u to v (possibly of length 0 if $u = v$) iff $(LFP_{R,x,y}\varphi)(u,v)$ holds. In any finite graph G, for any $k \geq 1$, $(\varphi^{\mathsf{G}})^k(\varnothing) = \{(x, y) |$ there is a path from x to y of length $\leq k-1\}$, and since the distance (the shortest length of a path) from a vertex to another vertex connected to it in G is at most $n-1$ if $||G|| = n$, the fixed point is obtained at most at $k = n$ i.e. after n iterations of the function φ^{G} on $\varnothing$.

$FO(LFP)$ is defined to be the logic obtained by adding the least fixed-point operator (LFP) to first order logic. Neil Immerman proved that a class of ordered finite structures is definable in $FO(LFP)$ if and only if it is decidable by a deterministic polynomial-time Turing machine (i.e. it is in the complexity class P) [5],[3]. This showed the importance of $FO(LFP)$ in descriptive complexity. The depth of an R-positive formula $\varphi(R, x_1, ..., x_k)$ in a finite structure A of size n, in symbols $|\varphi^{\mathsf{A}}|$, is defined to be the minimum r such that $(\varphi^{\mathsf{A}})^r(\emptyset) = (\varphi^{\mathsf{A}})^{r+1}(\emptyset)$ (this r is always less than or equal to n^k). The depth of φ as a function of n is defined by $|\varphi|(n) = max_{||A||=n}\{|\varphi^A|\}$ [3]. For example, the depth of φ in the example above is n. $IND[f(n)]$ is the sub-logic of $FO(LFP)$ in which only fixed points of first-order formulas φ for which $|\varphi|$ is $O[f(n)]$ are included. Note that, $FO(LFP) = \bigcup_{k=1}^{\infty} IND[n^k]$)[3]. The problem is to investigate the power of the depth of first-order formulas in defining relations inductively as least fixed points. In particular, the problem is to investigate the strictness of $IND[n] \subseteq IND[n^2] \subseteq IND[n^3] \subseteq \cdots$.

The Different Versions of the Problem

In this section we exhibit different versions of the problem. We begin by introducing some definitions and theorems, from [3], necessary for showing the equivalence of the different versions.

Lemma 1 *Every R-positive formula $\varphi(R, x_1, ..., x_k)$ is equivalent to a formula of the form $(Q_1 z_1, M_1) \dots (Q_s z_s, M_s)(\exists x_1 \dots \exists x_k, M_{s+1}) R(x_1, \dots, x_k)$ where the Q_i 's are quantifiers, the M_i 's are quantifier free formulas in which R does not occur, and $(\forall x, M)\psi$ means $\forall x(M \to \psi)$, and $(\exists x, M)\psi$ means $\exists x(M \wedge \psi)$.*

Proof. **cf. [3]**

We write QB to denote the quantifier block $(Q_1 z_1, M_1)...(Q_s z_s, M_s)(\exists x_1 ... x_k, M_{s+1})$. Thus for any finite structure A and any $r \in \mathbb{N}$, $(\varphi^A)^r(\emptyset) = \{\bar{a} \in A^k : A \vDash [QB]^r \boldsymbol{false}[\bar{a}]\}$, here $[QB]^r$ means QB literally repeated r times. It follows immediately that if $t = |\varphi|(n)$ and A is any structure of size n then
$A \vDash ((LFP_{R,\bar{x}}\varphi)\bar{y} \leftrightarrow [QB]^t \boldsymbol{false}(\bar{y}))[\bar{a}]$ for all $\bar{a} \in A^k$. [3]

Definition 2.1 *$FO[t(n)]$ is defined to be the class of properties definable by quantifier blocks iterated $O[t(n)]$ times [3]. A class $S \subseteq STRUC[\tau]$ (where τ is a finite vocabulary and $STRUC[\tau]$ is the class of all finite τ-structures) is a member of*

$FO[t(n)]$ *if and only if there exist quantifier free formulas* $M_i, 0 \le i \le s$, *a quantifier block* $QB = (Q_1 x_1, M_1)...(Q_s x_s, M_s)$, *a tuple of constants* $\bar{c}$, *and a function* $f(n) = O[t(n)]$, *such that for all* $\mathcal{A} \in STRUC[\tau]$, $\mathcal{A} \in S \Leftrightarrow \mathcal{A} \vDash \left([QB]^{f(||\mathcal{A}||)} M_0\right)(\bar{c}/\bar{x})$.

Thus Lemma1 implies that $IND[t(n)] \subseteq FO[t(n)]$ for all $t(n)$, i.e. for every class S of finite τ-structures (for any finite vocabulary τ), if S is definable in $IND[t(n)]$ then $S \in FO[t(n)]$.

Definition 2.2 *We say that a function* $s : \mathbb{N} \to \mathbb{N}$ *is time constructible iff there is a deterministic Turing machine running in time* $O[s(n)]$ *that on input* 0^n, *i.e.,* n *in unary, computes* $s(n)$ *in binary.*

Lemma 2 *For any polynomially bounded* $t(n)$ *and every class* S *of finite* τ*-structures (for any finite vocabulary* τ *), if* S *is decidable in parallel time* $t(n)$ *then* S *is definable in* $IND[t(n)]$.

Proof. cf. [3] for the proof and the definition of parallel time computation.

Lemma 3 *For every polynomially bounded parallel time constructible* $t(n)$ *and every class* S *of finite* τ*-structures (for any finite vocabulary* τ *) , if* $S \in FO[t(n)]$ *then* S *is decidable in parallel time* $t(n)$.

Proof. cf. [3]

Theorem 1 *For every polynomially bounded parallel time constructible* $t(n)$ *and every class* S *of finite* τ*-structures (for any finite vocabulary* τ *) the following are equivalent : 1.* S *is decidable in parallel time* $t(n)$*. 2.* S *is definable in* $IND[t(n)]$*. 3.* $S \in FO[t(n)]$.

Proof. Follows directly from Lemmas 1, 2, and 3.

Thus the question of the strictness of the sequence $IND[n] \subseteq IND[n^2] \subseteq IND[n^3] \subseteq ...$ is equivalent to the questions of the strictness of the following two sequences $FO[n] \subseteq FO[n^2] \subseteq FO[n^3] \subseteq ...$
$CRAM[n] \subseteq CRAM[n^2] \subseteq CRAM[n^3] \subseteq ...$ where $CRAM[t(n)]$ is the class of problems decidable in parallel time $t(n)$ with the kind of parallel time computation introduced in chapter 5 of [3].
There is also another equivalent version of the question in computational complexity, a one related to circuit complexity:

Definition 2.3 *A boolean circuit is a directed, acyclic graph,* $\mathrm{C} = (V, E, G_\wedge, G_\vee, G_\neg, I, r) \in STRUC[\tau_c]$ *with* $\tau_c = \langle E, G_\wedge, G_\vee, G_\neg, I, r\rangle$ *where* E *is of arity* 2 *and represents the edge relation,* $G_\wedge$ *is of arity one and consists of the internal vertices that are and-gates,* $G_\vee$ *is of arity one and consists of the internal vertices that are or-gates,* $G_\neg$ *is of arity one and consists of the internal vertices that are not-gates, and* I *is of arity one and consists of the leaves that are on i.e. carry the value* 1*, where a leaf is a vertex with no edges entering it.* r *is a constant symbol that represents the root of the tree.*

Definition 2.4 *A query is any mapping* $I : STRUC[\sigma] \to STRUC[\tau]$ *from the finite structures of one vocabulary to the finite structures of another vocabulary, that is polynomially bounded. That is, there is a polynomial* p *such that for all* $\mathrm{A} \in STRUC[\sigma]$, $\|I(A)\| \le p(\|A\|)$.
A boolean query is a map $I_b : STRUC[\sigma] \to \{0,1\}$. *A boolean query may also be thought of as a subset of* $STRUC[\sigma]$ *- the set of structures* A *for which* $I(\mathrm{A}) = 1$.

Definition 2.5 *(First-Order Queries) Let* σ *and* τ *be any two vocabularies where* $\tau = \langle R_1, ..., R_r, c_1, ..., c_s\rangle$ *and each* R_i *has arity* a_i*, and let* k *be a fixed natural number. A first-order query is a map*

$$I : STRUC[\sigma] \to STRUC[\tau]$$

defined by an $r+s+1$*-tuple of first-order formulas,* $\varphi_0, \varphi_1, ..., \varphi_r, \psi_1, ..., \psi_s$, from $FO[\sigma]$. *For each structure* $\mathrm{A} \in STRUC[\sigma]$ *these formulas describe a structure* $I(\mathrm{A}) \in STRUC[\tau]$,

$$I(\mathrm{A}) = \langle |I(\mathrm{A})|, R_1^{I(\mathrm{A})}, ..., R_r^{I(\mathrm{A})}, c_1^{I(\mathrm{A})}, ..., c_s^{I(\mathrm{A})}\rangle$$

The universe of $I(\mathrm{A})$ *is a first-order definable subset of* A^k ,

$$|I(A)| = \{< b^1, \dots, b^k > \in A^k : A \vDash \varphi_0(b^1, \dots, b^k)\}$$

Each relation $R_i^{I(\mathrm{A})}$ *is a first-order definable subset of* $|I(\mathrm{A})|^{a_i}$,

$$R_i^{I(A)} = \{(< b_1^1, \dots, b_1^k >, \dots, < b_{a_i}^1, \dots, b_{a_i}^k >) \in |I(A)|^{a_i} : A \vDash \varphi_i(b_1^1, \dots, b_{a_i}^k)\}$$

Each constant symbol $c_j^{I(\mathrm{A})}$ *is a first-order definable element of* $|I(\mathrm{A})|$,

$c_j^{I(\mathrm{A})} =$ *the unique* $\langle b^1, ..., b^k\rangle \in |I(\mathrm{A})|$ *such that* $A \vDash \Psi_j(b^1, \dots, b^k)$

A first-order query is either boolean, and thus defined by a first-order sentence, or is a k-ary first-order query, for some k.

Definition 2.6 *Let* C *be a sequence of circuits* $\mathsf{C} = \{C_i \mid i = 1,2,...\}$. *Let* $I : STRUC[\tau_s] \to STRUC[\tau_c]$ *be a query such that for all* $n \in \mathsf{N}$, $I(0^n) = C_n$, *where* $\tau_s = \langle \leq, S \rangle$ *is the vocabulary of binary strings. That is, on input a string of* n *zeros the query produces circuit* n. *If* I *is a first order query, then* C *is a first-order uniform sequence of circuits.*

Definition 2.7 *(**Circuit Complexity**) Let* $t(n)$ *be a polynomially bounded function and let* $S \subseteq STRUC[\tau]$ *be a boolean query. Then* S *is in the (first-order uniform) circuit complexity class* $AC[t(n)]$ *iff there exists a first-order query* $I : STRUC[\tau_s] \to STRUC[\tau_c]$ *defining a uniform class of circuits* $C_n = I(0^n)$ *with the following properties:*

1. *For all* $\mathsf{A} \in STRUC[\tau]$,
   $A \in S \Leftrightarrow C_{||A||}$ *accepts* A.
2. *The depth of* C_n *is* $O[t(n)]$.
3. *The gates of* C_n *consist of unbounded fan-in "and" and "or" gates.*

Theorem 2 *For all polynomially bounded first-order constructible* $t(n)$, *the following classes are equal:* $CRAM[t(n)] = IND[t(n)] = FO[t(n)] = AC[t(n)]$

Proof. cf.[3]

Thus our question is also equivalent to the question of the strictness of $AC[n] \subseteq AC[n^2] \subseteq AC[n^3] \subseteq ...$

What we suggest

We expect the sequence to be strict and our expectation is motivated by a well-known theorem from computational complexity, namely, the time hierarchy theorem for deterministic Turing machines [6],[7], which states that if f, g are time-constructible functions satisfying $f(n)\log(f(n)) = o[g(n)]$, then $DTIME(f(n)) \subsetneq DTIME(g(n))$, i.e. the class of queries decidable by $f(n)$-time deterministic Turing machines is strictly contained in the class of queries decidable by $g(n)$-time deterministic Turing machines (n^k and n^{k+1} satisfy the conditions of the theorem). From theorem 1 we know that the inductive depth equals parallel time i.e. the classes in $FO[t(n)]$ (or equally in $IND[t(n)]$) are precisely the classes decidable in parallel time $t(n)$, and since the (sequential) time hierarchy does not collapse, we expect that the parallel time hierarchy does not collapse. We introduce some definitions and facts before mentioning our suggestion.

Definition 3.1

($\mathrm{Q}(\mathrm{C})$, *the queries computable in* C*) Let* $I : STRUC[\sigma] \to STRUC[\tau]$ *be a query, and* C *a complexity class. We say that* I *is computable in* C *iff the boolean query* I_b *is an element of* C*, where* $I_b = \{(\mathrm{A}, i, a) \mid \text{The i-th bit of } bin(I(\mathrm{A})) \text{ is } "a"\}$*. And* $\mathrm{Q}(\mathrm{C})$ *is the set of all queries computable in* C*:* $\mathrm{Q}(\mathrm{C}) = \mathrm{C} \cup \{I \mid I_b \in \mathrm{C}\}$.

Definition 3.2 *(Many-One Reduction)*
Let C *be a complexity class, and let* $K \subseteq STRUC[\sigma]$ *and* $H \subseteq STRUC[\tau]$ *be boolean queries. Suppose that the query* $I : STRUC[\sigma] \to STRUC[\tau]$ *is an element of* $\mathrm{Q}(\mathrm{C})$ *with the property that for all* $\mathrm{A} \in STRUC[\sigma]$, $\mathrm{A} \in K \Leftrightarrow I(\mathrm{A}) \in H$ *Then* I *is called a* C*-many-one reduction from* K *to* H*. We say that* K *is* C*-many-one reducible to* H*, in symbols,* $K \leq_{\mathrm{C}} H$*. For example, when* I *is a first-order query, this is called a first-order reduction, in symbols* $\leq_{fo}$.

Definition 3.3 *Let* K *be a boolean query, Let* C *be a complexity class. We say that* K *is* C*-complete under first-order reductions if* 1. $K \in \mathrm{C}$, and, 2. for all $H \in \mathrm{C}$, $H \leq_{fo} K$.

Definition 3.4
(Alternating Reachability)
Let an alternating graph $G = (V, E, A, s, t)$ *be a directed graph whose vertices are labeled universal or existential.* $A \subseteq V$ *is the set of universal vertices. Let* $\tau_{ag} = \langle E, A, s, t \rangle$ *be the vocabulary of alternating graphs. Let* $P_a^G(x, y)$ *be the smallest relation on vertices of* G *such that: 1.* $P_a^G(x, x)$ *2. If* x *is existential and* $P_a^G(z, y)$ *holds for some edge* (x, z) *then* $P_a^G(x, y)$*. 3. If* x *is universal, and there is at least one edge leaving* x*, and* $P_a^G(z, y)$ *holds for all edges* (x, z) *then* $P_a^G(x, y)$*.* $REACH_a = \{G \mid P_a^G(s, t)\}$

It can be easily seen that $REACH_a$ is definable in $IND[n]$

Theorem 3 $REACH_a$ *is* P*-complete under first-order reductions.*

Proof. cf. [3]

Since there are problems, such as alternating reachability, which are in $IND[n]$ and are P-complete under first-order reductions, it follows that if $IND[n^k]$ - for some k - is closed under first-order reductions then $P = IND[n^k]$ and the hierarchy collapses at the

k-th level. On the other hand if for every k, $IND[n^k]$ is not closed under first-order reductions then $P \neq IND[n^k]$ for every k and the hierarchy does not collapse, but this does not necessarily mean that the sequence is strict. We suggest tackling the problem by investigating whether $IND[n^k]$ are closed under first-order reductions.

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