Planar Wilson Loops for Deriving Interquark Potential in Lattice QCD

Physics

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Abstract

Calculation techniques of Lattice QCD (Quantum Chromodynamics) offer special opportunities for the treatment of nonperturbative effects of strong interaction in physics. In Lattice QCD the static quark-anti-quark potential can be derive from the Wilson loops. In this paper we use planar Wilson loops to determine the static quark-anti-quark potential. Using simulation with SU(3) gauge field configuration we derive quark-anti-quark potential for different values of coupling constant and for different lattice sizes. Calculations are made for 100 configurations, statistically independent, of gauge fields of the lattice. Finally, we extrapolate for different lattice constant to take string tension parameter.

1. Introduction

Wilson loops were introduced in the 1970s (Creutz and Moriarty, 1982), as an attempt at a non-perturbative formulation of quantum chromodynamics (QCD) (Gattringer and Lang, 2009), or at least as a convenient collection of variables for dealing with the strongly-interacting regime of QCD. The use of Wilson loops to measure the string tension in non-abelian gauge theories is the simplest framework in which non-perturbative quantities can be computed (at least numerically) with good precision. The problem of confinement, which Wilson loops (Wilson, 1974), were designed to solve, remains unsolved to this day. The measurements on the lattice of the string tension and of the potential, both related to Wilson loop measures, have parallel histories initiated for the potential from Schilling and Bali (1993), Deldar (2000), Bali (2000), Parisi et al. (1983) which confirm the existence of the linear potential for the intermediate distances for the fundamental and higher representations in SU(3). Based on these measurements, quarks are confined in all representations of SU(3). Motivated by these important physical results, in this paper we measure the effective quark potential from planar Wilson loops using high parallel calculation techniques for different lattices volume. We have used the standard methods to compute or "measure" the quark-anti-quark potential. The quark-anti-quark potential can be extracted by large time behavior of the Wilson loops. We start from a closed rectangular path C(R,T) with extension *R x T*. The potential is extracted from so called Wilson loops W(R,T), which are defined to be the trace of path-ordered products of link variables $U_\mu(n)$ along the path C (R,T). This loop construction corresponds to the world lines of a quark-anti-quark pair at rest, separated by distance R from each other and "travelling" over time separation T. In Euclidian time, this observable will reveal the static "*ground state"* energy for large T values. To calculate the energy one must compute the correlation function of quark anti-quark operator for different time:

$$W(R,T) = \langle 0|O_R(0)O_R(T)^*|0\rangle = \langle O_R(0)O_R(T)^*\rangle \qquad (1)$$

when operator $O_R(t)$ is gauge invariant and is defined as:

$$O_R(t) = \bar{q}(t,0)U\big((t,0) \to (t,R)\big)q(t,R) \qquad (2)$$

and $U\big((t,0) \to (t,R)\big)$ is gauge field that connect static quarks from point (t,0) to point (t, R). Finally the Wilson loops can be defined as:

$$W(R,T) = \langle tr\, U\big((0,0) \to (0,R)\big)U\big((0,R) \to (T,R)\big)U\big((T,0) \to (T,R)\big)^* U\big((0,0) \to (T,0)\big)^* \rangle \qquad (3)$$

The Wilson loops can be written as:

$$W(R,T) = \sum_{n\geq 1} c_n e^{-V_n(R)T} \qquad (4)$$

Where the ground state $V_1(R) \equiv V(R)$ and other values for n>1 are the potentials of the excited states. So we can extract potential assuming:

$$W(R,T) \cong c_1 e^{-V(R)T}, \qquad (5)$$

and finally calculate the effective potentials from:

$$V(R)_{eff} = -\log\frac{W(R,T+1)}{W(R,T)} \qquad (6)$$

Finally, for different R we can fit the effective potential according to theoretical model in physical unit:

$$V(R)_{eff} = V_0 + KR + \frac{\alpha}{R} \qquad (7)$$

where α is a constant (the coefficient of the Coulomb-like term) and K the string tension parameter. We can take the representation of this model in lattice unit by multiplying with lattice parameter the equation (7):

$$\hat{V}(R)_{eff} = \hat{V}_0 + \hat{K}\frac{R}{a} + \frac{a}{R}\hat{\alpha}. \qquad (8)$$

1.1. The computation of string tension parameter

Monte Carlo calculations of lattice QCD in the quenched approximation have reached considerable precision. For example, the accuracy of hadron mass calculations using the Wilson action is quoted to be well below 1% at finite values of the lattice spacing (Aoki S. et al., 1998) and the running of the coupling is known to a precision of around 1% over energy scales varying by two orders of magnitude (Luscher et al., 1994), (Capitani et al., 1998). In comparison to such a level of accuracy, a low energy reference scale in the Yang-Mills theory (Hooft G., 2005) is known with much worse precision, despite the fact that such a scale is very important for the

analysis of the results (Edwards et al., 1997). For such purposes was introduced a reference scale, r_0(Sommer, 1994). This length scale is defined in terms of the force, *F(R)*, between external static charges in the fundamental representation. It is the solution of $r_0^2 F(r_0) = 1.65$. The constant on the right hand side was chosen such that r_0 has a value of approximately 0.5 *fm* in QCD (Sommer, 1994). The computation of string tension of static quarks has an important role in non-perturbative methods of QCD, because it is a physical quantity which is accurately determined from experiment (phenomenological bosonic string) and is easily computed on the lattice. In order to take dimensional results one has to compute the lattice spacing *a*, which need a physical quantity as a reference, such as string tension. We have calculated the string tension for 100 configuration, for different lattice sizes 8^4, 12^4 and 16^4.

The calculation of the string tension is made by fitting the lattice potential in equation (8). The value of R taking for fit is on range 0.5-8. The calculated string tension parameter in lattice unit, $\tilde{K}$, is related to the empirical string tension $K = (440MeV)^2$(see Appendix I), by $\tilde{K} = a^2 K$ (comparing the equations (7) and (8)). We have calculated lattice spacing a according to Guagnelli (1998). The physical volume of the lattice (L^4) with length L is related to the lattice volume (N^4) with N point for direction by L= aN. In order to take physical length L of the lattice constant we changed the value of coupling constant $\beta = 6/g^2$ (g is the QCD coupling) in each simulation with different lattice volume (8^4, 12^4, 16^4). The changed values of β are taken from parameterization of the low - energy reference scale r_0:

$$ln\left[\frac{a}{r_0}\right] = -1.6805 - 1.7139(\beta - 6) + 0.8155(\beta - 6)^2 - 0.6667(\beta - 6)^3 \qquad (9)$$

according to Guagnelli (1998).

2. Parallel Computations Using FermiQCD

Lattice Quantum Chromo-dynamics (LQCD) is an algorithmic formulation of QCD, the mathematical model that describes quarks and their interactions. Quarks are the basic components of most of the known matter, including protons and neutrons. LQCD is based on complex numerical algorithms derived from a mathematical analogy between "Feynman paths" in Quantum Mechanics and "steps" in the Markov Chain Monte Carlo algorithm. LQCD is a large scale computing project where computations are typically very expensive and run on dedicated supercomputers and large computer clusters for many months. Numerical lattice QCD calculations using Monte Carlo methods can be extremely computationally intensive, requiring the use of the largest available supercomputers. To reduce the computational burden, the so-called quenched approximation can be used, in which the quark fields are treated as non-dynamic "frozen" variables. In this paper we use a special software called Fermiqcd (Di Pierro et al., 2004), (Di Pierro, 2001), (Di Pierro, 2002) which is a collection of classes, functions and parallel algorithms for lattice QCD written in C++. It's a library that includes C++ methods for matrix manipulation, advanced statistical analysis (such as Jackknife and Boostrap) and optimized algorithms for interprocess communications of distributed lattices and fields. These communications are implemented using Message Passing Interface (MPI) but MPI calls are *hidden* to the high level

algorithms that constitute Fermiqcd. Fermiqcd works also on single processor computers and, in this case, MPI is not required. In this paper we work with several processors to increase the speed of data processing. One of the main differences between Fermiqcd and libraries developed by other collaborations is that it follows an *object oriented design* as opposed to a procedural design. Fermiqcd should not be identified exclusively with the implementation of the algorithms but, rather, with the strict specifications that define its Application Program Interface. The objects of the language include complex numbers (*mdp complex*), matrices (*mdp matrix*), lattices (*mdp lattice*), fields (gauge field, fermi field, staggered field), propagators (fermi propagator) and actions. Some of the advantages of using this software for our study are the following:

- Programs written in Fermiqcd are easy to write, read and modify since the Fermiqcd syntax resembles the mathematical syntax used in Quantum Field Theory articles and books.
- Programs are portable in the sense that they can, in principle, be compiled with any ANSI C++ compiler. Hardware specific optimizations are coded in the library and hidden from the high level programmer.
- The high level programmer does not have to deal with parallelization issues since the underlying objects deal with it. Fermiqcd communications are based on MPI.
- Programs are easier to debug because the usage of Fermiqcd objects and algorithms does not require explicit use of pointers. All memory management is done by the objects themselves.

In this work the calculations are performed in one of the clusters for supercomputing of HP-SEE (High-Performance Computing Infrastructure for South East Europe's Research Communities) project, that is located in Bulgaria (BG HPC). We first tested the Fermiqcd software deployment on small parallel machines and allocation of computer time and simulation and than on large parallel machines. The Fermiqcd code was ported for MPI usage before our usage. We test it for different lattice's sizes. We started from the example that estimated a single Wilson loop and adapted to calculate 36 rectangular Wilson loops for different sizes of time (T=1,…,6) and (R=1,…,6) direction on a lattice. The output file with all Wilson loops for 100 configurations was than used in Matlab for statistical study of physical quantity of interest like fitting quark – anti-quark potential, find the errors of coefficients that modeling the potential with Jackknife method, etc. The algorithm of this code follows this steps that are written below:

Step 1: Include Fermiqcd libraries

Step 2: Start (*mdp.open_wormholes(argc,argv*))

Step 3: Define this parameters: Lattice volume; SU(n) gauge group; Number of gauge configurations or number of Monte Carlo steps; Coupling constant (beta) that sets lattice spacing

Step 4: make: A 4D lattice (*mdp_lattice lattice(4,L)*); A gauge field U (*gauge_field U(lattice,n)*); A random gauge configuration (*set_hot(U)*)

Step 5: Loop over the Monte Carlo steps or over number of configuration

Step 6: Make a generic path on lattice to construct the shape of Wilson loop

Step 7: Loop over all possible paths in $\mu\nu$ plane, for $\mu = 0,1,2,3$ and $\nu = 0,1,2,3$

Step 8: Calculate the real part of the trace of path-ordered products of link variables $U_\mu(n)$ along the path, so the Wilson loops

Step 9: Save the Wilson loops in a *.dat* file format.

Step 10: Close with (*mdp.close_wormholes()*)

We made quenched simulation with SU(3) gauge field configuration with Wilson fermions for 100 configuration.

3. Results and Discussion

The simulation are made with Wilson action on 8^4, 12^4, 16^4 lattices at three lattice spacing, for a total number of 100 SU(3) gauge configurations, carried out on the BG HPC cluster, using Fermiqcd software. As we expected, we find that the quark-anti-quark potential is dominated by a Coulomb–like term at short distances with an *R* dependent coupling $V(R) \cong V_0 + \alpha/R$ for *R* small and by a linearly rising term at large distances $V(R) \cong V_0 + KR$ for *R* large, that confirm the fact of *confinement* of quarks within hadrons as in Schilling and Bali (1993), Deldar (2000), Bali (2000), Parisi et al. (1983)

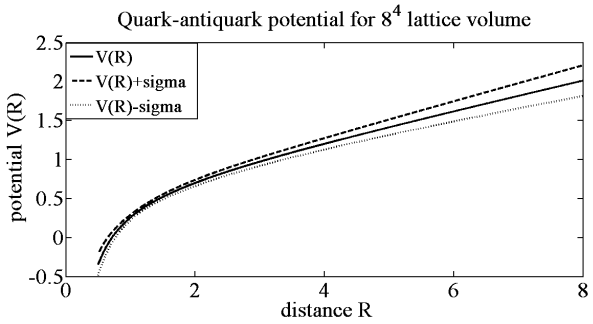

Figure 3. Fitting curve of quark-anti-quark potential for beta=5.7 with the band of errors, for 8^4 lattice volume, in lattice unit

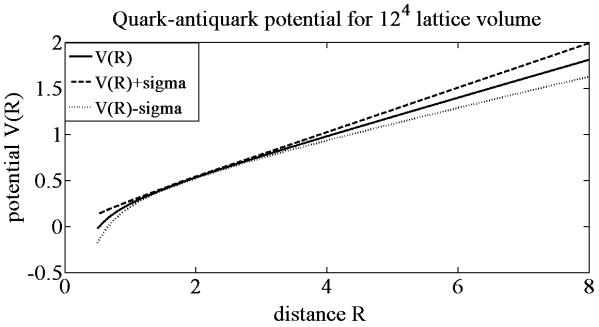

Figure 2. Fitting curve of quark-anti-quark potential for beta=5.85 with the band of errors, for 12^4 lattice volume, in lattice unit

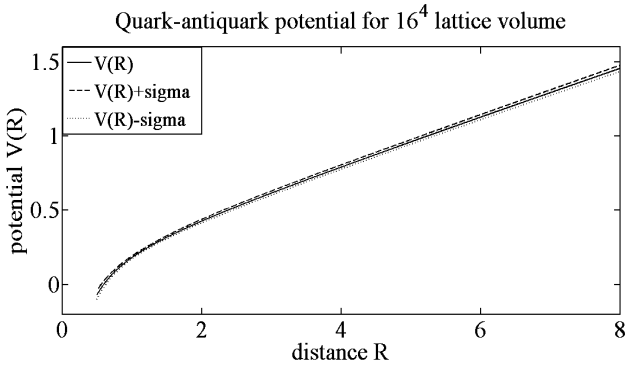

Figure 3. Fitting curve of quark-anti-quark potential for beta=6 with the band of errors, for 16^4 lattice volume, in lattice unit

Evaluation of statistical errors for coefficients of the model is made using jackknife method (Young, 2009), (Miller, 1974). The Jackknife method as a non-parametric method is used more when you have to calculate statistical errors of derived quantity The results are presented in Table 1.

Table 2 The values of lattice spacing *a*, string tension $\hat{K}$, with respective errors for different lattice volume

Lattice volume	Coupling constant	Lattice spacing *a* (in *fm*)	String tension (in lattice unit)	Statistical Error of *a*	Statistical Error of
8^4	5.7	0.2023(08)	0.1868(84)	1.0536e-04	3.77e-02
12^4	5.85	0.1846(21)	0.2023(64)	9.0614e-05	3.87e-02
16^4	6	0.1248(91)	0.1609(73)	1.1363e-05	3.91e-03

As we can see from Table 1, the values of lattice spacing and string tension parameter for different lattice volume are calculated within the range of statistical error. For example for lattice 8^4 the lattice spacing is a = [0.2023(08) ± 1.0536x10^{-4}] (*fm*) and = [0.1868(84) ± 3.77x10^{-2}] (in lattice unit). In our paper we follow the same idea as Deldar (2000), when are calculated these parameters bur for small values of , equal to 1.7, 2.4 and 3.1. We take approximately the same results as Schilling and Bali (1993), for = 5.7 using high parallel calculation techniques.

Our first objective was to test these new techniques calculating the static quark-anti-quark potential and string tension parameter. We have to take in consideration that we have to include much more Wilson loops in order to take better results. The string tension "measurement" should be performed at larger values of R. Practically, this requires large lattice volumes and huge statistics in the later works. The parallel computation techniques with Fermiqcd will be the most important feature to get better results. We tested the standard method of computing quark-anti-quark potential from Wilson loops using parallel computation techniques with Fermiqcd. We found that for number of processors *np=4*, we get saturation, the simulation time doesn't change. In the continues study we will implement *volume Wilson loops* to extract quark-anti-quark potential, in order to improve our results.

Appendix I

The model of Bosonic string

One of the main goals of a non-perturbative formulation of Gauge Theories is trying to understand the phenomenon of quark confinement. In a non-Abelian gauge theory color sources are confined in color-singlets, which cannot be separated. At a qualitative level such confinement is understood by assuming that the color electric fluxes emanating from the quarks are squeezed into a string-like configuration (figure 4). Such a gauge field configuration has a constant energy density per unit length, i.e. E = KR. This energy density K is called *the string tension*, and it is the fundamental physical quantity for a pure gauge field. According to this model the quarks are supposed to be connected with "strings" which represent flux color of quarks.

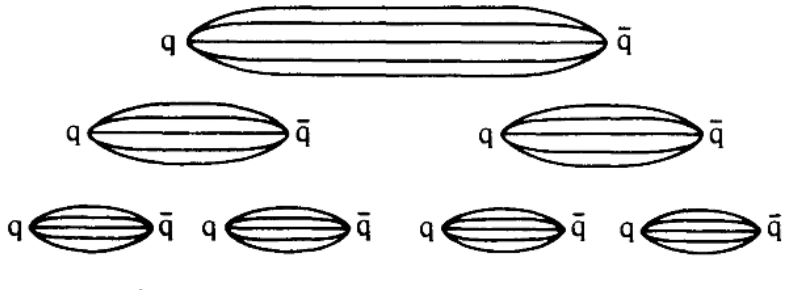

Figure 4. The schematic representation of quark-anti-quark flux color tubes.

Production of a quark-anti-quark pair by vacuum or by annihilations of pair $e+e^-$ leads to the cutting of string as its length R increase. This makes the initial string to create two new strings and so on. This process stop until the length of strings will be of hadronic scale i.e. 1*fm* ($200MeV^{-1}$), or until the energy will be of hadron masses range i.e ~ 1GeV.

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